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Hilbert schemes of points on the minimal resolution and soliton equations

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Abstract. The equivariant and ordinary cohomology rings of Hilbert schemes of points on the minimal resolution $\mathbb{C}^2//\Gamma$ for cyclic Γ are studied using vertex operator technique, and connections between these rings and the class algebras of wreath products are explicitly established. We further show that certain generating functions of equivariant intersection numbers on the Hilbert schemes and related moduli spaces of sheaves on $\mathbb{C}^2//\Gamma$ are τ-functions of 2-Toda hierarchies.

1. Introduction

In the past few years, the theory of vertex operators (cf. [**FLM**]) has found remarkable applications in the study of the Hilbert schemes $X^{[n]}$ of points on a surface X (see [**Na2, Le, LQW1, LQW2, Ru, Vas, Wa1**] and the references therein). To a large extent, the construction of Heisenberg algebra by Nakajima [**Na1**] (also cf. [**Gro**]) was a key starting point. It is well known that the Hilbert-Chow morphism from the Hilbert scheme $X^{[n]}$ to the symmetric product X^n/S_n is a resolution of singularities. An isomorphism has been established between the cohomology rings of the Hilbert schemes and the Chen-Ruan orbifold cohomology rings of the symmetric products for the affine plane $\mathbb{C}^2$ in [**LS, Vas**] via different approaches, and for a large class of quasi-projective surfaces in [**LQW1**]. These and other results have supported Ruan's general conjectures [**Ru**] on the relations between the orbifold cohomology ring of an orbifold and the cohomology ring of its crepant resolution. On the other hand, a certain generating function of equivariant intersection numbers on the Hilbert schemes $(\mathbb{C}^2)^{[n]}$ are shown [**LQW2**] to be τ-functions of the 2-Toda hierarchies.

When $X = \mathbb{C}^2//\Gamma$ is the minimal resolution of $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $SL_2(\mathbb{C})$, we have the following resolution of singularities [**Wa1**]

$$\pi_n : (\mathbb{C}^2//\Gamma)^{[n]} \longrightarrow \mathbb{C}^{2n}/\Gamma_n$$

obtained by the composition $(\mathbb{C}^2//\Gamma)^{[n]} \to (\mathbb{C}^2//\Gamma)^n/S_n \to (\mathbb{C}^2/\Gamma)^n/S_n \cong \mathbb{C}^{2n}/\Gamma_n$. Here $\Gamma_n := \Gamma^n \rtimes S_n$ is the wreath product. When Γ is cyclic, we fix a distinguished $T = \mathbb{C}^*$ action on $\mathbb{C}^2//\Gamma$ which induces an action on $(\mathbb{C}^2//\Gamma)^{[n]}$ with isolated fixed

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points. Throughout the paper, we will assume that Γ is a cyclic finite subgroup of $SL_2(\mathbb{C})$. For example, if Γ is of order 2, then $\mathbb{C}^2//\Gamma$ is isomorphic to the total space of the cotangent bundle over the projective line $\mathbb{P}^1$.

A main goal of this paper is to study the (equivariant) cohomology ring of Hilbert schemes of points on $\mathbb{C}^2//\Gamma$ using vertex operator technique and to develop its relations to soliton equations. This study specializes when Γ is trivial to the study of [**Vas, LQW2**] in the affine plane case. In addition, we shall establish Ruan's conjecture on the cohomology ring isomorphism for the resolution $\pi_n : (\mathbb{C}^2//\Gamma)^{[n]} \to \mathbb{C}^{2n}/\Gamma_n$ and provide an explicit map for such an isomorphism.

Let us now discuss the paper in more details. First of all, generalizing the affine plane case studied by Vasserot [**Vas**], we introduce a ring structure on

$$\mathbb{H}_n = H_T^{2n}((\mathbb{C}^2//\Gamma)^{[n]})$$

which encodes the equivariant cohomology ring structure of $H_T^*((\mathbb{C}^2//\Gamma)^{[n]})$. We further construct a Heisenberg algebra acting irreducibly on

$$\mathbb{H} = \bigoplus_{n=0}^{\infty} \mathbb{H}_n.$$

This is an equivariant analog of the Heisenberg algebra in [**Na1, Na2**] (also cf. [**Vas**]). Our study uses in an essential way a very concrete and useful description of $\mathbb{C}^2//\Gamma$ and its torus action provided by Ito-Nakamura [**IN**].

Next, we introduce an explicit map from the ring $\mathbb{H}_n$ to the class algebra $R(\Gamma_n)$ and show that it is a ring isomorphism. This result specializes to a result in [**Vas**] in the affine plane case. Denote by $\mathcal{G}_\Gamma^*(n)$ the graded ring associated to a natural filtration on $R(\Gamma_n)$ (cf. [**Wa3**]). We obtain an explicit map from $H^*((\mathbb{C}^2//\Gamma)^{[n]})$ to $\mathcal{G}_\Gamma^*(n)$ which is further shown to be also a graded ring isomorphism. When Γ is trivial, this has been established in [**LS, Vas**] and in [**LQW1**] using different methods. With the help of the Heisenberg algebra in the setup of wreath products [**FJW, Wa2**] and the Heisenberg algebra on Hilbert schemes constructed above, our maps further identify several distinguished linear bases on both sides. Our above isomorphism of graded rings establishes Ruan's conjecture for the crepant resolution $\pi_n : (\mathbb{C}^2//\Gamma)^{[n]} \to \mathbb{C}^{2n}/\Gamma_n$, since one can identify $\mathcal{G}_\Gamma^*(n)$ as the Chen-Ruan orbifold cohomology ring of $\mathbb{C}^2/\Gamma_n$. The resolution π_n seems to be the first nontrivial example beyond the Hilbert-Chow morphism where Ruan's conjecture is established in a constructive way (see [**EG**] for an earlier nonconstructive approach toward Ruan's conjecture). A direct and constructive ring isomorphism from $H^*((\mathbb{C}^2//\Gamma)^{[n]})$ to $\mathcal{G}_\Gamma^*(n)$ for non-cyclic Γ has yet to be constructed, though $\mathcal{G}_\Gamma^*(n)$ can still be defined for a general Γ, cf. [**EG, Wa3**].

In addition, we introduce a family of moduli spaces of sheaves on $\mathbb{C}^2//\Gamma$, which are isomorphic to the Hilbert schemes $(\mathbb{C}^2//\Gamma)^{[n]}$ and parameterized by certain integral lattice in $H_T^2(\mathbb{C}^2//\Gamma)$. Using an operator approach, we study the T-equivariant Chern characters of some distinguished T-equivariant tautological bundles over these moduli spaces. We identify the Chern character operators, which are defined in terms of the T-equivariant Chern characters, with some familiar operators acting on the fermionic Fock space associated to the integral lattice in $H_T^2(\mathbb{C}^2//\Gamma)$. We then formulate generating functions of the equivariant intersection numbers of these Chern characters, and recast them in an operator formalism. It follows from

standard arguments that they are τ-functions of the 2-Toda hierarchies of Ueno-Takasaki [**UT**]. These results generalize the affine plane case studied in [**LQW2**].

This paper is organized as follows. In Section 2, we study the torus-equivariant geometry of the Hilbert schemes and formulate the Heisenberg algebra. In Section 3, we established the two ring isomorphisms in the equivariant and ordinary cohomology setups. In Section 4, we introduce a certain moduli space of sheaves on the minimal resolution $\mathbb{C}^2//\Gamma$ and study the related fermionic Fock space. In Section 5, we show that certain generating functions of the equivariant intersection numbers on these moduli spaces are τ-functions.

2. Hilbert schemes of points on the minimal resolution

2.1. The minimal resolution. Let $\Gamma = \langle a \rangle$ be the cyclic group of order r. Below we often regard Γ as the subgroup of $SL_2(\mathbb{C})$ which consists of the diagonal matrices $a^i = \mathrm{diag}(\varepsilon^i, \varepsilon^{-i}), 0 \le i < r$, where $\varepsilon = e^{2\pi\sqrt{-1}/r}$ is a primitive r-th root of unity. We denote by $\Gamma^* = \{\gamma_0, \gamma_1, \ldots, \gamma_{r-1}\}$ the set of complex irreducible characters of Γ, with the character table of Γ given by

$$\gamma_k(a^i) = \varepsilon^{ik}, \qquad 0 \le i, k \le r-1.$$

In other words, the character table is given by the $r \times r$ matrix

$$M = [\varepsilon^{ik}]_{0 \le i,k \le r-1}.$$

Let $\pi : \mathbb{C}^2//\Gamma \to \mathbb{C}^2/\Gamma$ be the minimal desingularization of $\mathbb{C}^2/\Gamma$, and let $X = \mathbb{C}^2//\Gamma$. The exceptional fiber $\pi^{-1}(0)$ consists of $(r-1)$ projective lines $\Sigma_1, \ldots, \Sigma_{r-1}$, with the configuration of a Dynkin diagram

$$A_{r-1}: \qquad \begin{array}{ccccccc} \circ & \text{---} & \circ & \text{---} & \circ \;\cdots & \text{---} & \circ \\ \Sigma_1 & & \Sigma_2 & & & & \Sigma_{r-1} \end{array}$$

in the following sense: Σ_i and Σ_j intersect if and only if $|i-j| = 1$; and when this is so they intersect transversally at one point. Following Ito-Nakamura [**IN**], we can identify X with the subvariety of $(\mathbb{C}^2)^{[r]}$ which consists of the points corresponding to the following Γ-invariant ideals in the coordinate ring $\mathbb{C}[z_1, z_2]$ of $\mathbb{C}^2$:

$$I(O) \;:=\; \prod_{(a_1,a_2)\in O} \mathfrak{m}_{(a_1,a_2)} = (z_1^r - a_1^r, z_1 z_2 - a_1 a_2, z_2^r - a_2^r), \tag{2.1}$$

$$I_i(p_i : q_i) \;:=\; (p_i z_1^i - q_i z_2^{r-i}, z_1 z_2, z_1^{i+1}, z_2^{r+1-i}) \tag{2.2}$$

where $\mathfrak{m}_{(a_1,a_2)} = (z_1 - a_1, z_2 - a_2)$, O stands for a Γ-orbit in $\mathbb{C}^2$ disjoint from the origin, $1 \le i \le r-1$, and $(p_i : q_i) \in \mathbb{P}^1$. For each $1 \le i \le r-1$, the rational curve Σ_i consists of all the points in $(\mathbb{C}^2)^{[r]}$ corresponding to the ideals (2.2).

Moreover, X admits an affine open cover $\{X_i\}_{0 \le i \le r-1}$:

$$X = \cup_{0 \le i \le r-1} X_i, \quad \text{where } X_i \cong \mathrm{Spec}\ \mathbb{C}[z_{i,1}, z_{i,2}].$$

The inclusion map $X_i \hookrightarrow X$ is given by the morphism defined via the universal property of $(\mathbb{C}^2)^{[r]}$ from the following 2-dimensional flat family of Γ-invariant ideals in $\mathbb{C}[z_1, z_2]$:

$$\mathfrak{I}_i(a_{i,1}, a_{i,2}) := (z_1^{i+1} - a_{i,1} z_2^{r-i-1}, z_1 z_2 - a_{i,1} a_{i,2}, z_2^{r-i} - a_{i,2} z_1^i) \tag{2.3}$$

where $(a_{i,1}, a_{i,2})$ stands for points in $X_i \cong \mathbb{C}^2$. Note that when $a_{i,1}a_{i,2} \neq 0$, we have $\mathfrak{I}_i(a_{i,1}, a_{i,2}) = I(O)$ where O is the Γ-orbit of the point $(a_1, a_2) \in \mathbb{C}^2$ with

$$a_1 = a_{i,1}^{(r-i)/r} a_{i,2}^{(r-i-1)/r}, \qquad a_2 = a_{i,1}a_{i,2}/a_1 = a_{i,1}^{i/r} a_{i,2}^{(i+1)/r}$$

(i.e., $a_{i,1} = a_1^{i+1}/a_2^{r-i-1}$ and $a_{i,2} = a_2^{r-i}/a_1^i$). Similarly, we obtain

$$\mathfrak{I}_i(a_{i,1}, 0) = I_{i+1}(1 : a_{i,1}), \qquad 0 \le i \le r-2 \tag{2.4}$$

$$\mathfrak{I}_{r-1}(a_{r-1,1}, 0) = (z_1^r - a_{r-1,1}, z_2), \tag{2.5}$$

$$\mathfrak{I}_i(0, a_{i,2}) = I_i(a_{i,2} : 1), \qquad 1 \le i \le r-1 \tag{2.6}$$

$$\mathfrak{I}_0(0, a_{0,2}) = (z_1, z_2^r - a_{0,2}). \tag{2.7}$$

Let $\xi_0, \ldots, \xi_{r-1} \in X$ be the points corresponding respectively to the ideals:

$$I_1(1:0),\ I_1(0:1) = I_2(1:0), \ldots,\ I_{r-2}(0:1) = I_{r-1}(1:0),\ I_{r-1}(0:1). \tag{2.8}$$

Then $\xi_0 \in \Sigma_1$, $\{\xi_1\} = \Sigma_1 \cap \Sigma_2, \ldots, \{\xi_{r-2}\} = \Sigma_{r-2} \cap \Sigma_{r-1}$, and $\xi_{r-1} \in \Sigma_{r-1}$. Moreover, we see from (2.4)–(2.7) that ξ_i is the origin of the open affine chart X_i, $\Sigma_{i+1} - \{\xi_{i+1}\}$ is the $z_{i,1}$-axis of X_i when $0 \le i \le r-2$, and $\Sigma_i - \{\xi_{i-1}\}$ is the $z_{i,2}$-axis of X_i when $1 \le i \le r-1$. Let Σ_0 be the $z_{0,2}$-axis of X_0, and Σ_r be the $z_{r-1,1}$-axis of X_{r-1}.

2.2. Fixed point classes and bilinear forms. Let $T = \mathbb{C}^*$ act on $\mathbb{C}^2$ by

$$s(z_1, z_2) = (sz_1, s^{-1}z_2), \quad s \in T. \tag{2.9}$$

This action induces a T-action on $(\mathbb{C}^2)^{[r]}$ which preserves the minimal resolution $X = \mathbb{C}^2//\Gamma$ as a subvariety of $(\mathbb{C}^2)^{[r]}$. In view of (2.1) and (2.2), the locus X^T of T-fixed points is

$$X^T = \{\xi_0, \ldots, \xi_{r-1}\}.$$

Since the family of ideals in (2.3) is T-invariant, X_i is T-invariant. In addition, T acts on points $(a_{i,1}, a_{i,2}) \in X_i$ by $s(a_{i,1}, a_{i,2}) = (s^{-r}a_{i,1}, s^r a_{i,2})$, i.e., T acts on the coordinate functions $(z_{i,1}, z_{i,2})$ of X_i by

$$s(z_{i,1}, z_{i,2}) = (s^r z_{i,1}, s^{-r} z_{i,2}). \tag{2.10}$$

The T-action on X induces a T-action on the Hilbert scheme $X^{[n]}$, which again has isolated fixed points. As explained below, these isolated fixed points ξ_λ are parametrized by the multi-partitions λ in the finite set

$$\mathcal{P}_n(r) = \{\lambda = (\lambda^0, \cdots, \lambda^{r-1}) \mid |\lambda^0| + \cdots + |\lambda^{r-1}| = n\}. \tag{2.11}$$

More explicitly, a T-fixed subscheme Z of X of length-n is a disjoint union of T-fixed subschemes Z_i, $i = 0, \ldots, r-1$, such that Z_i is supported at ξ_i, and $\sum_i \ell(Z_i) = n$. Put $n_i = \ell(\xi_i)$. Since $\xi_i \in X_i \cong \mathbb{C}^2$ is the origin and T acts on X_i by (2.10), we see from [**ES**] that the T-fixed subschemes $Z_i \in (X_i)^{[n_i]} \subset X^{[n_i]}$ are in one-to-one correspondence with the partitions λ^i of n_i. We denote such Z_i by $\xi_i^{\lambda^i}$ and so Z is given by $\xi_\lambda = \sum_i \xi_i^{\lambda^i}$ when $\lambda = (\lambda^0, \cdots, \lambda^{r-1})$. It is known [**ES**] that the tangent space $T_{\xi_i^{\lambda^i}} X^{[n_i]}$ of $X^{[n_i]}$ at $\xi_i^{\lambda^i}$ as a T-module decomposes as

$$T_{\xi_i^{\lambda^i}} X^{[n_i]} = \bigoplus_{\square \in \lambda^i} \left(\theta^{rh(\square)} \bigoplus \theta^{-rh(\square)} \right)$$

where θ is the 1-dimensional standard module of T, and $\Box$ runs over the cells in the Young diagram corresponding to the partition λ^i, $h(\Box)$ is the hook number of a cell $\Box$. Hence as a T-module the tangent space of $X^{[n]}$ at ξ_λ decomposes as

$$T_{\xi_\lambda} X^{[n]} = \bigoplus_{i=0}^{r-1} \bigoplus_{\Box\in\lambda^i} \left(\theta^{rh(\Box)} \bigoplus \theta^{-rh(\Box)}\right). \tag{2.12}$$

Let $H_T^*(M)$ be the equivariant cohomology of a smooth variety M with $\mathbb{C}$-coefficient. Then $H_T^*(M)$ is a $\mathbb{C}[t]$-module if we identify $H_T^*(\mathrm{pt})$ and $\mathbb{C}[t]$, where t is an element of degree-2. Putting $h(\lambda) = \prod_{i=0}^{r-1}\prod_{\Box\in\lambda^i} h(\Box)$, we see from (2.12) that the equivariant Euler class of the tangent bundle at $\xi_\lambda \in X^{[n]}$ is

$$e_T(T_{\xi_\lambda} X^{[n]}) = (-1)^n r^{2n} t^{2n} \prod_{i=0}^{r-1} \prod_{\Box\in\lambda^i} h(\Box)^2 = (-1)^n r^{2n} t^{2n} h(\lambda)^2. \tag{2.13}$$

For $\lambda \in \mathcal{P}_n(r)$, let $i_\lambda : \xi_\lambda \to X^{[n]}$ be the inclusion map. Let $[\xi_\lambda] \in H_T^{4n}(X^{[n]})$ be the equivariant cohomology class corresponding to the fixed point ξ_λ. Then, $[\xi_\lambda] = i_\lambda^!(1_{\xi_\lambda})$ where 1_{ξ_λ} is the unity in the ring $H_T^*(\xi_\lambda)$ and $i_\lambda^!$ is the Gysin map. For λ and μ in $\mathcal{P}_n(r)$, we see from the projection formula and (2.13) that

$$[\xi_\lambda] \cup [\xi_\mu] = \delta_{\lambda,\mu} e_T(T_{\xi_\lambda} X^{[n]})[\xi_\lambda] = \delta_{\lambda,\mu} \cdot (-1)^n r^{2n} t^{2n} h(\lambda)^2 [\xi_\lambda]. \tag{2.14}$$

Denote $\iota_n = \bigoplus_{\lambda\in\mathcal{P}_n(r)} i_\lambda : (X^{[n]})^T \to X^{[n]}$, and let $\iota_n^! : H_T^*((X^{[n]})^T)' \to H_T^*(X^{[n]})'$ be the induced Gysin map where $H_T^*(\cdot)' = H_T^*(\cdot) \otimes_{\mathbb{C}[t]} \mathbb{C}(t)$. By the localization theorem, $\iota_n^!$ is an isomorphism. The inverse $(\iota_n^!)^{-1}$ is given by

$$\alpha \to \left(\frac{i_\lambda^*\alpha}{e_T(T_{\xi_\lambda} X^{[n]})}\right)_{\lambda\in\mathcal{P}_n(r)} = \left(\frac{i_\lambda^*\alpha}{(-1)^n r^{2n} t^{2n} h(\lambda)^2}\right)_{\lambda\in\mathcal{P}_n(r)}. \tag{2.15}$$

We define a bilinear form $\langle -, - \rangle$ on $H_T^*(X^{[n]})' \otimes_{\mathbb{C}(t)} H_T^*(X^{[n]})' \to \mathbb{C}(t)$ by

$$\langle \alpha, \beta \rangle = (-1)^n p_n^! (\iota_n^!)^{-1} (\alpha \cup \beta) \tag{2.16}$$

where p_n is the projection of the set $(X^{[n]})^T$ of T-fixed points to a point.

Set $\mathbb{H}_n = H_T^{2n}(X^{[n]})$. From the spectral sequence computation and the fact that $H^{2k}(X^{[n]}) = 0$ for $k > n$, we see that $H_T^{4n}(X^{[n]}) = t^n \cdot H_T^{2n}(X^{n]}) = t^n \mathbb{H}_n$. So there is an induced (commutative associative) ring structure $\star$ on $\mathbb{H}_n$ defined by:

$$t^n (x \star y) = x \cup y, \qquad x, y \in \mathbb{H}_n.$$

Associated to $[\xi_\lambda] \in H_T^{4n}(X^{[n]})$, we can define $[\lambda] \in \mathbb{H}_n$ for $\lambda \in \mathcal{P}_n(r)$ by

$$[\lambda] = (-1)^n r^{-n} h(\lambda)^{-1} t^{-n} [\xi_\lambda] \in \mathbb{H}_n. \tag{2.17}$$

To emphasis the case $n = 1$, we introduce for $0 \le i \le r-1$ the notation:

$$\diamondsuit_i = -r^{-1} t^{-1} [\xi_i].$$

It follows from (2.14) and (2.16) that for $\lambda, \mu \in \mathcal{P}_n(r)$, we have

$$\langle [\lambda], [\mu] \rangle = \delta_{\lambda,\mu}. \tag{2.18}$$

Combining with (2.15), we see that the $[\lambda]$'s with $\lambda \in \mathcal{P}_n(r)$ form a $\mathbb{C}$-linear basis of $\mathbb{H}_n$. So the restriction of the bilinear form (2.16) to $\mathbb{H}_n \times \mathbb{H}_n$ is a nondegenerate

$\mathbb{C}$-*valued* bilinear form on $\mathbb{H}_n$ which again will be denoted by $\langle -, - \rangle$. Let

$$\mathbb{H} = \bigoplus_{n=0}^{\infty} \mathbb{H}_n. \tag{2.19}$$

Then we have an induced non-degenerate bilinear form $\langle -, - \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$.

LEMMA 2.1. (i) *We have the following identifications:*

$$rt = \diamondsuit_0 + \cdots + \diamondsuit_{r-1}, \quad [\Sigma_i] = \diamondsuit_{i-1} - \diamondsuit_i, \ 1 \le i \le r-1.$$

(ii) *The bilinear form $\langle -, - \rangle$ on $\mathbb{H}_1$ is also given by*

$$\langle t, t \rangle = \frac{1}{r}, \quad \langle t, [\Sigma_i] \rangle = 0, \quad \langle [\Sigma_i], [\Sigma_j] \rangle = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. It suffices to verify (i) since (ii) follow from (i) and (2.18). Applying the localization theorem (2.15) to $\alpha = rt \in H_T^*(X)$ and $n = 1$ yields

$$rt = \sum_{i=0}^{r-1} \frac{rt[\xi_i]}{-r^2t^2} = \sum_{i=0}^{r-1} \diamondsuit_i.$$

Since $\mathfrak{I}_{i-1}(a_{i-1,1}, 0) = I_i(1 : a_{i-1,1})$ when $1 \le i \le r-1$, the torus T acts on the points $(p_i : q_i) \in \Sigma_i$ by $s(p_i : q_i) = (p_i : s^{-r} q_i)$. Note that ξ_{i-1} and ξ_i are the points $(1 : 0)$ and $(0 : 1)$ in Σ_i respectively. By the localization theorem, we have

$$1_{\Sigma_i} = \frac{[\xi_{i-1}]}{-rt} + \frac{[\xi_i]}{rt} = -r^{-1}t^{-1}[\xi_{i-1}] + r^{-1}t^{-1}[\xi_i] \in H_T^*(\Sigma_i)'.$$

It follows that $[\Sigma_i] = \diamondsuit_{i-1} - \diamondsuit_i \in H_T^*(X)$. This proves (i). $\square$

2.3. Heisenberg algebra. In this subsection, we shall generalize the construction of the Heisenberg algebra in [**Vas**] (also cf. [**Na2**]). Let i be a positive integer, and Y be a T-invariant closed curve in X. Define

$$Y_{n,i} = \{(\xi, \eta) \in X^{[n+i]} \times X^{[n]} \mid \eta \subset \xi, \ \mathrm{Supp}(I_\eta / I_\xi) = \{y\} \in Y\}. \tag{2.20}$$

Let p_1 and p_2 be the projections of $X^{[n+i]} \times X^{[n]}$ to the two factors respectively. We define the linear operator $\mathfrak{p}_{-i}([Y]) \in \mathrm{End}(\mathbb{H})$ by

$$\mathfrak{p}_{-i}([Y])(\alpha) = p_1^{!}(p_2^*\alpha \cup [Y_{n,i}]) \in \mathbb{H}_{n+i} \tag{2.21}$$

for $\alpha \in H_T^{2n}(X^{[n]})$. Note that the restriction of p_1 to $Y_{n,i}$ is proper. We define $\mathfrak{p}_i([Y]) \in \mathrm{End}(\mathbb{H})$ to be the adjoint operator of $\mathfrak{p}_{-i}([Y])$. Alternatively, letting p_2' be the projection of $(X^{[n]})^T \times X^{[n-i]}$ to $X^{[n-i]}$, we see that

$$\mathfrak{p}_i([Y])(\alpha) = (-1)^i \cdot (p_2')^{!} \big((\iota_n \times \mathrm{Id})^{!}\big)^{-1} (p_1^*\alpha \cup [Y_{n-i,i}]) \in \mathbb{H}_{n-i}$$

for $\alpha \in H_T^{2n}(X^{[n]})$. Finally, we also put $\mathfrak{p}_0([Y]) = 0$.

Recall that Σ_0 is the $z_{0,2}$-axis in $X_0 \cong \mathrm{Spec}\, \mathbb{C}[z_{0,1}, z_{0,2}]$, and Σ_r is the $z_{r-1,1}$-axis in $X_{r-1} \cong \mathrm{Spec}\, \mathbb{C}[z_{r-1,1}, z_{r-1,2}]$. Both Σ_0 and Σ_r are closed in X. As in Lemma 2.1 (i), using the localization theorem, we obtain

$$[\Sigma_0] = -\diamondsuit_0, \qquad [\Sigma_r] = \diamondsuit_{r-1}. \tag{2.22}$$

Since $\diamondsuit_0, \ldots, \diamondsuit_{r-1}$ form a linear basis of $\mathbb{H}_1 = H_T^2(X)$, we see from Lemma 2.1 (i) and (2.22) that $\mathbb{H}_1$ has two more linear bases:

$$\{[\Sigma_0], [\Sigma_1], \ldots, [\Sigma_{r-1}]\}, \qquad \{[\Sigma_1], [\Sigma_2], \ldots, [\Sigma_r]\}. \tag{2.23}$$

Using one of the two bases in (2.23), we extend by linearity on α to obtain the linear operator $\mathfrak{p}_m(\alpha) \in \mathrm{End}(\mathbb{H})$ for every $\alpha \in \mathbb{H}_1$.

THEOREM 2.2. *The linear operators $\mathfrak{p}_m(\alpha)$, where $m \in \mathbb{Z}$ and $\alpha \in \mathbb{H}_1 = H^2_T(X)$, satisfy the Heisenberg algebra commutation relation:*

$$[\mathfrak{p}_m(\alpha), \mathfrak{p}_n(\beta)] = m\delta_{m,-n}\langle \alpha, \beta\rangle \ \mathrm{Id}. \tag{2.24}$$

Furthermore, $\mathbb{H}$ is the Fock space (i.e. an irreducible module) of this Heisenberg algebra with highest weight vector $|0\rangle$ which denotes the unity in $\mathbb{H}_0 \subset H^*_T(X^{[0]})$.

PROOF. By (2.23), $\{[\Sigma_0], [\Sigma_1], \ldots, [\Sigma_{r-1}]\}$ is a basis of $\mathbb{H}_1$. Since $\mathfrak{p}_0([\Sigma_i]) = 0$ and $\mathfrak{p}_{-m}([\Sigma_i])$ is the adjoint of $\mathfrak{p}_m([\Sigma_i])$, to prove (2.24) it suffices to prove that

$$[\mathfrak{p}_m([\Sigma_i]), \mathfrak{p}_n([\Sigma_j])] = 0 \tag{2.25}$$

$$[\mathfrak{p}_m([\Sigma_i]), \mathfrak{p}_{-n}([\Sigma_j])] = m\delta_{m,n}\langle [\Sigma_i], [\Sigma_j]\rangle \ \mathrm{Id} \tag{2.26}$$

for $m, n > 0$ and $0 \le i, j \le r$. When $i \ne j$, Σ_i and Σ_j are either disjoint or intersect transversally at exactly one point. Following the argument in [**Na2, Vas**], we conclude that (2.25) and (2.26) hold when $i \ne j$. To handle the case $i = j$, we see from Lemma 2.1 (i) and (2.22) that $[\Sigma_i] = -\sum_{0\le s\le r, s\ne i}[\Sigma_s]$. Thus,

$$\begin{aligned}
[\mathfrak{p}_m([\Sigma_i]), \mathfrak{p}_n([\Sigma_i])] &= -\sum_{0\le s\le r, s\ne i} [\mathfrak{p}_m([\Sigma_i]), \mathfrak{p}_n([\Sigma_s])] = 0, \\
[\mathfrak{p}_m([\Sigma_i]), \mathfrak{p}_{-n}([\Sigma_i])] &= -\sum_{0\le s\le r, s\ne i} [\mathfrak{p}_m([\Sigma_i]), \mathfrak{p}_{-n}([\Sigma_s])] \\
&= m\delta_{m,n}\langle [\Sigma_i], [\Sigma_i]\rangle \ \mathrm{Id}.
\end{aligned}$$

This completes the proof of (2.25) and (2.26), and whence (2.24).

To prove the second statement in our theorem, recall that the classes $[\lambda]$, as λ runs over all multi-partitions in $\mathcal{P}_n(r)$, form a linear basis of $\mathbb{H}_n$. Therefore,

$$\sum_{n=0}^{\infty} \dim(\mathbb{H}_n)\ q^n = \frac{1}{\prod_{k=1}^{\infty}(1-q^k)^r}. \tag{2.27}$$

On the other hand, it is well known that the Fock space of the Heisenberg algebra is irreducible (thanks to the non-degeneracy of the bilinear form on $\mathbb{H}_1$), and its character is given by the right-hand-side of (2.27). Hence we can identify the space $\mathbb{H}$ with the Fock space of the Heisenberg algebra. $\square$

For latter purpose we make the following definitions. Let $\mu = (\mu_1, \mu_2, \cdots, \mu_\ell)$ be a partition of the integer $|\mu| = \mu_1 + \cdots + \mu_\ell$, where $\mu_1 \ge \cdots \ge \mu_\ell \ge 1$. We will also make use of another notation $\mu = (1^{m_1}2^{m_2}\cdots)$, where m_i is the number of parts in μ equal to i. The length $\ell(\mu)$ is the number ℓ or $m_1 + m_2 + \cdots$ in the second notation. For $0 \le i \le r-1$ and $m \in \mathbb{Z}$, we define

$$\mathfrak{p}_m(c^i) = \sum_{j=0}^{r-1} \varepsilon^{-ij}\mathfrak{p}_m(\diamondsuit_j).$$

For $\lambda = (\lambda^0, \cdots, \lambda^{r-1})$ with $\lambda^i = (1^{m_1(i)} 2^{m_2(i)} \ldots)$, define

$$\mathfrak{p}_{-\lambda} := \prod_{i=0}^{r-1} \mathfrak{p}_{-\lambda^i}(\diamondsuit_i)|0\rangle, \quad \mathfrak{p}_{-\lambda^i}(\diamondsuit_i) = \prod_{k\geq 1} \mathfrak{p}_{-k}(\diamondsuit_i)^{m_k(i)} \tag{2.28}$$

$$\mathfrak{p}'_{-\lambda} := \prod_{i=0}^{r-1} \prod_{k\geq 1} \mathfrak{p}_{-k}(c^i)^{m_k(i)}|0\rangle. \tag{2.29}$$

2.4. Equivariant bundles and equivariant Chern characters. We denote the universal codimension-2 subscheme of $X^{[n]} \times X$ by

$$\mathcal{Z}_n = \{(\xi, x) \in X^{[n]} \times X \mid x \in \mathrm{Supp}(\xi)\}.$$

For a line bundle L on X, let $L^{[n]}$ denote the tautological rank-n vector bundle $\pi_{1*}(\mathcal{O}_{\mathcal{Z}_n} \otimes \pi_2^* L)$ on $X^{[n]}$, where π_1 and π_2 denote the projections of $X^{[n]} \times X$ to the factors. When L is T-equivariant over X, $L^{[n]}$ is T-equivariant over $X^{[n]}$. This construction is actually valid for every (quasi-)projective surface (besides X).

Next, using [**IN**] we describe certain distinguished T-equivariant line bundles over X, which were earlier defined in [**GSV**] by a different and more complicated method. Recall that X is identified with the subvariety of $(\mathbb{C}^2)^{[r]}$ consisting of the points corresponding to the ideals I in (2.1) and (2.2) of $\mathbb{C}[z_1, z_2]$. For such an I, $\mathbb{C}[z_1, z_2]/I$ is isomorphic to the regular representation of Γ. Denote by $\mathcal{O}_{\mathbb{C}^2}^{[r]}$ the tautological rank-r bundle over the Hilbert scheme $(\mathbb{C}^2)^{[r]}$. Each fiber of the rank-r vector bundle $\big(\mathcal{O}_{\mathbb{C}^2}^{[r]}\big)|_X$ over X carries the structure of the regular representation of Γ. We obtain T-equivariant line bundles $L_0, L_1, \ldots, L_{r-1}$ over X by decomposing $\big(\mathcal{O}_{\mathbb{C}^2}^{[r]}\big)|_X$ according to the irreducible characters $\gamma_0, \gamma_1, \ldots, \gamma_{r-1}$ of Γ:

$$\big(\mathcal{O}_{\mathbb{C}^2}^{[r]}\big)|_X \cong \bigoplus_{k=0}^{r-1} \gamma_k \otimes L_k. \tag{2.30}$$

To understand the fiber $L_k|_{\xi_i}$ of L_k at the T-fixed point ξ_i $(0 \leq i \leq r-1)$, we recall from (2.8) that ξ_i corresponds to the ideal $(z_1^{i+1}, z_1 z_2, z_2^{r-i})$ in $\mathbb{C}[z_1, z_2]$. The fiber of $(\mathcal{O}_{\mathbb{C}^2})^{[r]}$ at a point $Z \in (\mathbb{C}^2)^{[r]}$ is canonically identified with $H^0(\mathcal{O}_Z)$. Hence the fiber of $\big(\mathcal{O}_{\mathbb{C}^2}^{[r]}\big)|_X$ at ξ_i is canonically identified as

$$H^0(\mathcal{O}_{\xi_i}) = \mathrm{Span}\ (1, z_1, \ldots, z_1^i, z_2, \ldots, z_2^{r-i-1}).$$

Combining this with the standard action of Γ on $\mathbb{C}^2$ and (2.30), we conclude that

$$L_k|_{\xi_i} = \begin{cases} \mathbb{C} \cdot z_2^{r-k} & \text{if } 0 \leq i < k, \\ \mathbb{C} \cdot z_1^k & \text{if } k \leq i \leq r-1. \end{cases} \tag{2.31}$$

Now we study the Chern character of the T-equivariant tautological rank-n vector bundle $L_k^{[n]}$ over $X^{[n]}$. We begin with the description of its fiber over a T-fixed point $\xi_\lambda \in X^{[n]}$ where $\lambda = (\lambda^0, \ldots, \lambda^{r-1}) \in \mathcal{P}_n(r)$. The fiber of $L_k^{[n]}$ at ξ_λ is

$$L_k^{[n]}|_{\xi_\lambda} \cong \bigoplus_{i=0}^{r-1} L_k^{[|\lambda^i|]}|_{\xi_i^{\lambda^i}} \cong \bigoplus_{i=0}^{r-1} \big(L_k|_{\xi_i}\big) \otimes H^0(\mathcal{O}_{\xi_i^{\lambda^i}}). \tag{2.32}$$

If we write the partition λ^i in terms of its parts as $\lambda^i = (\lambda^i_1, \lambda^i_2, \cdots, \lambda^i_\ell)$, then the $\mathbb{C}$-vector space $H^0(\mathcal{O}_{\xi_i^{\lambda^i}}) \subset H^0(\mathcal{O}_{X_i})$ has a linear basis

$$\left\{1, z_{i,2}, \ldots, z_{i,2}^{\lambda_1^i-1}, z_{i,1}, z_{i,1}z_{i,2}, \ldots, z_{i,1}z_{i,2}^{\lambda_2^i-1}, \ldots, z_{i,1}^{\ell-1}, z_{i,1}^{\ell-1}z_{i,2}, \ldots, z_{i,1}^{\ell-1}z_{i,2}^{\lambda_\ell^i-1}\right\}$$

where $z_{i,1}$ and $z_{i,2}$ are the coordinate functions of X_i. By (2.10), T acts on $z_{i,1}$ and $z_{i,2}$ by $s(z_{i,1}, z_{i,2}) = (s^r z_{i,1}, s^{-r} z_{i,2})$. So as a T-module,

$$H^0(\mathcal{O}_{\xi_i^{\lambda^i}}) \cong \bigoplus_{\square\in\lambda^i} \theta^{rc_\square}$$

where $c_\square$ is the content of the cell $\square$. By (2.9), (2.31) and (2.32), we have

$$L_k^{[n]}|_{\xi_\lambda} \cong \left(\bigoplus_{i=0}^{k-1}\bigoplus_{\square\in\lambda^i}\theta^{(k-r)+rc_\square}\right)\bigoplus\left(\bigoplus_{i=k}^{r-1}\bigoplus_{\square\in\lambda^i}\theta^{k+rc_\square}\right).$$

Let $\mathrm{ch}_m^T(L_k^{[n]})$ be the m-th T-equivariant Chern character of $L_k^{[n]}$. Then,

$$\mathrm{ch}_m^T(L_k^{[n]})|_{\xi_\lambda} = \frac{1}{m!}\left(\sum_{i=0}^{k-1}\sum_{\square\in\lambda^i}\big((k-r+rc_\square)t\big)^m + \sum_{i=k}^{r-1}\sum_{\square\in\lambda^i}\big((k+rc_\square)t\big)^m\right).$$

By the projection formula, we have in $H_T^*(X^{[n]})'$ that

$$\begin{aligned}\mathrm{ch}_m^T(L_k^{[n]}) \cup [\xi_\lambda] &= i_\lambda^! \left(\mathrm{ch}_m^T(L_k^{[n]})|_{\xi_\lambda}\right) \\ &= \frac{1}{m!}\left(\sum_{i=0}^{k-1}\sum_{\square\in\lambda^i}\big((k-r+rc_\square)t\big)^m + \sum_{i=k}^{r-1}\sum_{\square\in\lambda^i}\big((k+rc_\square)t\big)^m\right)[\xi_\lambda].\end{aligned} \tag{2.33}$$

Let $m \geq 0$ and $0 \leq k \leq r-1$. For a nonnegative integer n, denote

$$\mathrm{ch}_{k;m}^{[n]} = t^{n-m}\mathrm{ch}_m^T(L_k^{[n]}) \in \mathbb{H}_n. \tag{2.34}$$

We define a *Chern character operator* $\mathfrak{G}_k$ (respectively, $\mathfrak{G}_{k;m}$) in $\mathrm{End}(\mathbb{H})$ by sending $a \in \mathbb{H}_n$ to $a \star \sum_{m\geq 0}\mathrm{ch}_{k;m}^{[n]}$ (respectively, to $a \star \mathrm{ch}_{k;m}^{[n]}$) in $\mathbb{H}_n$ for each n. Similarly, we define an operator $\mathfrak{G}_k(z) \in \mathrm{End}(\mathbb{H})$ by sending $a \in \mathbb{H}_n$ to $a \star \sum_{m\geq 0} z^m \mathrm{ch}_{k;m}^{[n]}$ for each n, where z is a variable. By definition, we have

$$\mathfrak{G}_k(z) = \sum_{m\geq 0}\mathfrak{G}_{k;m}z^m.$$

PROPOSITION 2.3. *Let $\varsigma(z) = e^{z/2} - e^{-z/2}$. For a given $\lambda = (\lambda^0, \cdots, \lambda^{r-1}) \in \mathcal{P}_n(r)$, we write $\lambda^i = (\lambda^i_1, \lambda^i_2, \ldots)$ in terms of parts. Then, for $0 \leq k \leq r-1$, we have*

$$\begin{aligned}\mathfrak{G}_k(z)\cdot[\lambda] &= \frac{1}{\varsigma(rz)}\left(\sum_{i=0}^{k-1}e^{(k-r)z}\left(\sum_{j=1}^{\infty}e^{(\lambda_j^i-j+1/2)rz} - \frac{1}{\varsigma(rz)}\right)\right. \\ &\qquad \left. + \sum_{i=k}^{r-1}e^{kz}\left(\sum_{j=1}^{\infty}e^{(\lambda_j^i-j+1/2)rz} - \frac{1}{\varsigma(rz)}\right)\right)[\lambda].\end{aligned} \tag{2.35}$$

PROOF. Recall that the contents associated to a partition $\lambda^i = (\lambda_1^i, \lambda_2^i, \dots, \lambda_\ell^i)$ are: $1-j, 2-j, \dots, \lambda_j^i - j, 1 \le j \le \ell$. A simple algebra manipulation gives us that

$$\sum_{\square \in \lambda^i} e^{c_\square z} = \frac{1}{\varsigma(z)} \left(\sum_{j=1}^{\infty} e^{(\lambda_j^i - j + 1/2)z} - \frac{1}{\varsigma(z)} \right). \tag{2.36}$$

The formula (2.33) and the definition (2.17) imply that

$$\mathfrak{G}_k(z) \cdot [\lambda] = \left(\sum_{i=0}^{k-1} \sum_{\square \in \lambda^i} e^{(k-r+rc_\square)z} + \sum_{i=k}^{r-1} \sum_{\square \in \lambda^i} e^{(k+rc_\square)z} \right) [\lambda]$$

which is then easily reduced to (2.35) by using (2.36). □

3. The isomorphisms of rings

3.1. Wreath products and Heisenberg algebra. For a finite group G, we denote by $R(G)$ the space of class functions on G, with two distinguished linear bases: one given by the irreducible characters of G and the other by the characteristic functions of the conjugacy classes of G. We denote by $R(G)_{\mathbb{Z}}$ the integral combination of irreducible characters of G. The standard bilinear form on $R(G)$ is defined to be such that the irreducible characters form an orthonormal basis. In this way, $R(G)_{\mathbb{Z}}$ endowed with this bilinear form becomes an integral lattice in $R(G)$.

Recall Γ is the cyclic group of order r. Given a positive integer n, let $\Gamma^n = \Gamma \times \cdots \times \Gamma$ be the n-th direct product of Γ. The symmetric group S_n acts on Γ^n by permutations: $\sigma(g_1, \cdots, g_n) = (g_{\sigma^{-1}(1)}, \cdots, g_{\sigma^{-1}(n)})$. The wreath product of Γ with S_n is defined to be the semi-direct product

$$\Gamma_n = \{(g, \sigma) | g = (g_1, \cdots, g_n) \in \Gamma^n, \sigma \in S_n\}$$

with the multiplication $(g, \sigma) \cdot (h, \tau) = (g\, \sigma(h), \sigma\tau)$.

The conjugacy classes of Γ_n can be described in the following way. Let $x = (g, \sigma) \in \Gamma_n$, where $g = (g_1, \cdots, g_n) \in \Gamma^n$ and $\sigma \in S_n$. Write σ as a product of disjoint cycles. For each such cycle $y = (i_1 i_2 \cdots i_k)$ we associate a *cycle-product* $g_{i_k} g_{i_{k-1}} \cdots g_{i_1} \in \Gamma$. For each integer $i \ge 1$, the number of k-cycles in σ whose cycle-product equals a^i will be denoted by $m_k(i)$. Denote by λ^i the partition $(1^{m_1(i)} 2^{m_2(i)} \cdots)$. Then each element $x = (g, \sigma) \in \Gamma_n$ gives rise to a multi-partition

$$(\lambda^0, \cdots, \lambda^{r-1}) \in \mathcal{P}_n(r)$$

which will be referred to as the *type* of x. It is well known (cf. [**Mac**]) that two elements of Γ_n are conjugate in Γ_n if and only if they have the same type.

Given a partition $\mu = (1^{m_1} 2^{m_2} \dots)$, we define $z_\mu = \prod_{i \ge 1} i^{m_i} m_i!$. The order of the centralizer of an element $x = (g, \sigma) \in \Gamma_n$ of type $\lambda = (\lambda^0, \cdots, \lambda^{r-1})$ is

$$Z_\lambda = r^{\ell(\lambda)} \prod_{i=0}^{r-1} z_{\lambda^i}.$$

For $\mu \vdash m$, we associate with the irreducible character s_μ of the symmetric group S_m and its corresponding representation U_μ. Denote by V_i the irreducible representation of Γ associated to the character γ_i. Then the wreath product $\Gamma_m = \Gamma^m \rtimes S_m$ acts irreducibly on $U_\mu \otimes V_i^{\otimes m}$ where Γ^m acts on the second tensor factor only and S_m acts diagonally. We denote the associated character of Γ_m by $s_\mu(\gamma_i)$.

More generally, let $\lambda = (\lambda^0, \cdots, \lambda^{r-1})$ be a multi-partition with $\|\lambda\| := |\lambda^0| + \ldots + |\lambda^{r-1}| = n$. Naturally $W_\lambda = \bigotimes_i \left(U_{\lambda^i} \otimes V_i^{\otimes |\lambda^i|} \right)$ is a representation of $\prod_i \Gamma_{|\lambda^i|}$ which is a subgroup of Γ_n. The induced Γ_n-representation $\mathrm{Ind}_{\prod_i \Gamma_{|\lambda^i|}}^{\Gamma_n} W_\lambda$ is irreducible and its associated character is given by

$$s_\lambda = \prod_i s_{\lambda^i}(\gamma_i). \tag{3.1}$$

Letting d_{λ^i} be the degree of s_{λ^i} (i.e. the dimension of the corresponding representation) and $h(\lambda) = \prod_i h(\lambda^i)$ where $h(\lambda^i)$ is the hook product associated to λ^i, we see that the degree d_λ of the character s_λ is

$$d_\lambda = \frac{n!}{\prod_i |\lambda^i|!} \prod_i d_{\lambda^i} = \frac{n!}{\prod_i h(\lambda^i)} = \frac{|\Gamma_n|}{r^n h(\lambda)}. \tag{3.2}$$

We define

$$\mathbb{R}_\Gamma = \bigoplus_{n=0}^{\infty} R(\Gamma_n).$$

It carries a bilinear form, denoted by $\langle -, - \rangle$, induced from the standard one on each $R(\Gamma_n)$. There is an action on $\mathbb{R}_\Gamma$ (cf. [**FJW, Wa1**]) of a Heisenberg algebra which is generated by $\mathfrak{a}_n(\gamma_i), n \in \mathbb{Z}, 0 \le i \le r-1$ with the commutation relation:

$$[\mathfrak{a}_m(\gamma_i), \mathfrak{a}_n(\gamma_j)] = m\delta_{m,-n}\delta_{i,j}\mathrm{Id}. \tag{3.3}$$

Here Id denotes the identity operator on $\mathbb{R}_\Gamma$. We further extend the definition of $\mathfrak{a}_n(\gamma)$ to all $\gamma \in R(\Gamma)$ by linearity. The operators $\mathfrak{a}_n(\gamma)$ (respectively, $\mathfrak{a}_{-n}(\gamma)$) for $n > 0$ have been defined in terms of restriction functors (respectively, induction functors) and will be referred to as annihilation operators (respectively, creation operators). It is known (cf. [**FJW**]) by (3.1) and the formulation of the Heisenberg operators that, for $m > 0$ and $\lambda = (\lambda^0, \cdots, \lambda^{r-1})$,

$$\begin{aligned} \mathfrak{a}_{\pm m}(\gamma_j)(s_{\lambda^i}(\gamma_i)) &= 0, \qquad j \neq i && (3.4) \\ \mathfrak{a}_{-m}(\gamma_i)(s_{\lambda^i}(\gamma_i)) &= \sum_{|\mu| = m + |\lambda^i|} c(m, \lambda^i; \mu)\, s_\mu(\gamma_i) && (3.5) \end{aligned}$$

where $c(m, \lambda^i; \mu) \in \mathbb{Q}$ depends only on m and the partitions λ^i, μ. It follows that

$$\mathfrak{a}_{-m}(\gamma_i)(s_\lambda) = \sum_{|\mu| = m + |\lambda^i|} c(m, \lambda^i; \mu)\, s_{(\lambda^0, \cdots, \lambda^{i-1}, \mu, \lambda^{i+1}, \cdots, \lambda^{r-1})}. \tag{3.6}$$

In particular, the coefficients $c(m, \lambda^i; \mu)$'s do not depend on r, i.e. they are the same structure constants as for the symmetric group case.

Denote by c^i the class function in $R(\Gamma)$ which takes value r on the element a^i and 0 elsewhere. Following [**FJW**], we can use the character table of Γ and linearity to define another set of generators $\mathfrak{a}_m(c^i), 0 \le i \le r-1$, of the Heisenberg algebra by

$$\mathfrak{a}_m(c^i) = \sum_{k=0}^{r-1} \varepsilon^{-ik} \mathfrak{a}_m(\gamma_k).$$

We have

$$[\mathfrak{a}_m(c^i), \mathfrak{a}_n(c^j)] = rm\, \delta_{m,-n}\delta_{i+j,r}\ \mathrm{Id}.$$

Now the space $\mathbb{R}_\Gamma$ is an irreducible module over this Heisenberg algebra. Denote by $|0\rangle$ the unity in $R(\Gamma_0)$. Then we have

$$\mathbb{R}_\Gamma = \mathbb{C}[\mathfrak{a}_{-n}(c^i) \mid n > 0, 0 \le i \le r-1] \cdot |0\rangle.$$

Given $\lambda = (\lambda^0, \ldots, \lambda^{r-1})$ where $\lambda^i = (1^{m_1(i)} 2^{m_2(i)} \ldots)$ and $\|\lambda\| = n$, we define

$$\begin{aligned} \mathfrak{a}_{-\lambda} &= \prod_{i=0}^{r-1} \prod_{k \ge 1} \mathfrak{a}_{-k}(c^i)^{m_k(i)} |0\rangle; \\ \mathfrak{a}^R_{-\lambda} &= \prod_{i=0}^{r-1} \prod_{k \ge 1} \mathfrak{a}_{-k}(\gamma_i)^{m_k(i)} |0\rangle. \end{aligned}$$

REMARK 3.1. We can show that $\mathfrak{a}_{-\lambda}$ is the class function on Γ_n which takes value Z_λ on the conjugacy class of type λ, and 0 elsewhere.

We have three distinguished linear bases for $R(\Gamma_n)$:

$$\{\mathfrak{a}_{-\lambda}\}, \{\mathfrak{a}^R_{-\lambda}\}, \{s_\lambda\}, \qquad \lambda \in \mathcal{P}_n(r).$$

The transition matrix between the bases $\{\mathfrak{a}^R_{-\lambda}\}$ and $\{s_\lambda\}$ is given by the r-tuple version of the character table matrix of the symmetric groups. The transition matrix between the bases $\{\mathfrak{a}_{-\lambda}\}$ and $\{s_\lambda\}$ is precisely the character table of Γ_n.

3.2. The class algebras of wreath products. For any given finite group G, the space $R(G)$ of complex-valued class function carries an algebra structure (which is often refereed to as the *class algebra* of G) given by the *convolution product*:

$$(\beta \circ \gamma)(x) = \sum_{y \in G} \beta(xy^{-1})\gamma(y), \quad \beta, \gamma \in R(G), \ x \in G.$$

It is well known that, for two irreducible characters β and γ of G,

$$\beta \circ \gamma = \delta_{\beta,\gamma} \frac{|G|}{d_\gamma} \, \gamma, \tag{3.7}$$

where d_γ is the *degree* of the irreducible character γ. In this paper, we mainly apply these considerations to the group Γ_n.

Let x be an element of Γ_n of type $\lambda = (\lambda^0, \cdots, \lambda^{r-1}) \in \mathcal{P}_n(r)$. We define the *modified type* of x to be $\widetilde{\lambda} \in \mathcal{P}_{n-\ell}(r)$, where $\ell = \ell(\lambda^0)$, as follows: $\widetilde{\lambda}^i = \lambda^i$ for $i > 0$ and $\widetilde{\lambda}^0 = (\lambda^0_1 - 1, \cdots, \lambda^0_\ell - 1)$ if we write the partition $\lambda^0 = (\lambda^0_1, \cdots, \lambda^0_\ell)$. The *degree* $\|x\|$ of x is defined to be $\|\widetilde{\lambda}\|$, which equals $\|\lambda\| - \ell(\lambda^0)$. Apparently, two conjugate elements have the same degree, and it makes sense to talk about the degree of a conjugacy class of Γ_n. It was shown [**Wa3**] that

$$\|xy\| \le \|x\| + \|y\|, \qquad x, y \in \Gamma_n$$

and thus the degree defines a ring filtration for $R(\Gamma_n)$. Also it was shown that this notion of degree coincides with the fermionic degree number or age introduced by Zaslow and Ito-Reid. Denote the associated graded ring by

$$\mathcal{G}_\Gamma(n) = \bigoplus_{i=1}^{n} \mathcal{G}^i_\Gamma(n)$$

where $\mathcal{G}^i_\Gamma(n)$ consists of elements in $R(\Gamma_n)$ of degree i.

Among the conjugacy classes of Γ_n, the conjugacy class $K_{[21^{n-2}]}$ of "transposition" plays a special role. We decompose $K_{[21^{n-2}]}$ into a disjoint union:

$$K_{[21^{n-2}]} = \bigsqcup_{j=1}^{n} K_{[21^{n-2}]}(j), \qquad K_{[21^{n-2}]}(j) = \bigsqcup_{1\le i<j} K_{[21^{n-2}]}(i,j)$$

where $K_{[21^{n-2}]}(i,j)$ consists of elements $((g_1,\dots,g_n),(i,j)) \in \Gamma_n$, where all g_k except g_i and g_j are equal to $1\in\Gamma$, and $g_j = g_i^{-1}$ runs over Γ.

The elements $M_j = M_{j;n} = \sum_{x\in K_{[21^{n-2}]}(j)} x,\ 1\le j\le n$, in the group algebra $\mathbb{C}[\Gamma_n]$ were introduced in [**Wa2**], and they satisfy several favorable properties, e.g. $M_iM_j = M_jM_i$ for all i,j. Note that $M_1 = 0$. We call them the Jucys-Murphy (JM) elements for Γ_n since they reduce to the usual Jucys-Murphy elements of the symmetric group S_n when Γ is trivial.

Let $\Gamma^{(i)}$ denote the i-th copy of Γ in Γ_n. Given $\alpha\in R(\Gamma)$, we denote $\alpha^{(i)}$ to be the copy of α in $\mathbb{C}[\Gamma^{(i)}]\subset\mathbb{C}[\Gamma_n]$. Letting z be a formal parameter, we define

$$\Xi_n^m(\alpha) = \frac{1}{r^m m!}\sum_{i=1}^{n} {M_i}^m \circ \alpha^{(i)}, \qquad \Xi_n^{(\alpha)}(z) = \sum_{m\ge 0} \Xi_n^m(\alpha)\, z^m.$$

A key property of $\Xi_n^m(\alpha)$ is that $\Xi_n^m(\alpha)$ lies in $R(\Gamma_n)$. We further define the operator $\mathfrak{O}^m(\alpha)\in \mathrm{End}(\mathbb{R}_\Gamma)$ (respectively, $\mathfrak{O}^{(\alpha)}(z)$) to be the convolution product with $\Xi_n^m(\alpha)$ (respectively, $\Xi_n^{(\alpha)}(z)$) in $R(\Gamma_n)$ for every $n\ge 0$.

Let us introduce the following vertex operator associated to $\gamma_i, 0\le i\le r-1$:

$$V(\gamma_i;w,q) \;=\; \exp\left(\sum_{k>0}\frac{(q^k-1)w^k}{k}\mathfrak{a}_{-k}(\gamma_i)\right)\exp\left(\sum_{k>0}\frac{(1-q^{-k})w^{-k}}{k}\mathfrak{a}_{k}(\gamma_i)\right).$$

Write $V(\gamma_i;w,q) = \sum_{m\in\mathbb{Z}} V_m(\gamma_i;q)w^{-m}$. It is established in [**Wa2**][1] that

$$\mathfrak{O}^{(\gamma_i)}(z) = \frac{e^{rz}}{(e^{rz}-1)^2}\left(V_0(\gamma_i;e^{rz})-1\right), \quad 0\le i\le r-1. \tag{3.8}$$

When Γ is trivial, this specializes to a result of [**LT**].

PROPOSITION 3.2. *For $\lambda = (\lambda^1,\cdots,\lambda^{r-1})\in\mathcal{P}_n(r)$ with $\lambda^i = (\lambda^i_1,\lambda^i_2,\dots)$, we have*

$$\mathfrak{O}^{(\gamma_i)}(z)\cdot s_\lambda = \frac{1}{\varsigma(rz)}\left(\sum_{j=1}^{\infty} e^{(\lambda^i_j - j+1/2)rz} - \frac{1}{\varsigma(rz)}\right)\, s_\lambda. \tag{3.9}$$

PROOF. First let $r=1$. It is known that the eigenvalues of the JM elements of the symmetric group S_n on an irreducible module V_ν (where ν is a partition of n) are given by the contents of the Young tableaux associated to ν, cf. [**LT, LQW2**]. Then a simple calculation using generating function gives us (3.9). By (3.8),

$$\frac{e^z\left(V_0(1;e^z)-1\right)}{(e^z-1)^2}\cdot s_\nu = \frac{1}{\varsigma(z)}\left(\sum_{j=1}^{\infty} e^{(\nu_j-j+1/2)z} - \frac{1}{\varsigma(z)}\right)\, s_\nu. \tag{3.10}$$

Recall that $s_\lambda = \prod_j s_{\lambda^j}(\gamma_j)$. So by applying (3.4), (3.8) and (3.10) (with z replaced by rz, ν by λ^i, etc) to the γ_i-component, we obtain (3.9). □

[1]Our convention on $\mathfrak{O}^{(\alpha)}(z)$ here differs from [**Wa2**] by a scaling of z by a factor r.

3.3. An isomorphism in the equivariant setup. Recall the well-known fact that the Hilbert-Chow morphism $\tau_n : X^{[n]} \to X^n/S_n$, which sends an element of $X^{[n]}$ to its support, is a resolution of singularities. Combining with the minimal resolution $\pi : X \to \mathbb{C}^2/\Gamma$, we have the following commutative diagram

$$\begin{array}{ccc} X^{[n]} & \longrightarrow & X^n/S_n \\ \big\downarrow & & \big\downarrow \\ \mathbb{C}^{2n}/\Gamma_n & \xleftarrow{\cong} & (\mathbb{C}^2/\Gamma)^n/S_n \end{array}$$

which give us a (crepant) resolution of singularities $\pi_n : X^{[n]} \to \mathbb{C}^{2n}/\Gamma_n$. This was the starting point on why Hilbert schemes have something to do with wreath products (cf. [**Wa1**] and the references therein).

We define a linear map $\phi : \mathbb{H}_n \longrightarrow R(\Gamma_n)$ by letting

$$\phi([\lambda]) = s_\lambda \tag{3.11}$$

for each $\lambda \in \mathcal{P}_n(r)$, which is clearly an isomorphism of vector spaces. Putting these isomorphisms together, we obtain a linear isomorphism $\phi : \mathbb{H} \to \mathbb{R}_\Gamma$.

Lemma 3.3. *The linear isomorphism $\phi : \mathbb{H} \to \mathbb{R}_\Gamma$ commutes with the action of the Heisenberg creation operators. More precisely, for $m > 0$, we have*

$$\phi\big(\mathfrak{p}_{-m}([\Sigma_i]) \cdot [\lambda]\big) = \begin{cases} -\mathfrak{a}_{-m}(\gamma_0) \cdot s_\lambda & \text{if } \ i = 0, \\ \mathfrak{a}_{-m}(\gamma_{i-1} - \gamma_i) \cdot s_\lambda & \text{if } \ 1 \le i \le r-1, \\ \mathfrak{a}_{-m}(\gamma_{r-1}) \cdot s_\lambda & \text{if } \ i = r. \end{cases}$$

Proof. When $r = 1$, T acts on $\mathbb{C}^2$ by $s(z_1, z_2) = (sz_1, s^{-1}z_2)$. In this case, Lemma 3.3 follows from the results in [**Vas**], i.e., given a partition λ, we have

$$-\mathfrak{p}_{-m}([\Sigma_0])[\lambda] \quad = \sum_{|\mu|=m+|\lambda|} c(m, \lambda; \mu)\, [\mu], \tag{3.12}$$

where the coefficients $c(m, \lambda; \mu)$ were defined in (3.5). We note that (3.12) remains to be valid if the T-action on $\mathbb{C}^2$ is replaced by $s(z_1, z_2) = (s^k z_1, s^{-k} z_2)$ with $k > 0$ and $[\lambda]$ is defined to be $h(\lambda)^{-1}(-kt)^{-|\lambda|}[\xi_\lambda]$ as in (2.17).

Assume $r = 2$. Let $\lambda = (\lambda^0, \lambda^1) \in \mathcal{P}_n(2)$. Recall that $\Sigma_0 \subset X_0$ and that ξ_0 is the only T-fixed point on Σ_0 and X_0. Now the inclusion map $\iota_0 : X_0 \to X$ is T-equivariant. It induces the T-equivariant inclusion map $\iota_{01} : X_0^{[\tilde{n}+m]} \to X^{[n+m]}$ sending ξ to $\xi + \xi_1^{\lambda^1}$, where $\tilde{n} = |\lambda^0|$. Similarly, we have T-equivariant morphisms $\iota_{02} : X^{[n+m]} \times X_0^{[\tilde{n}]} \to X^{[n+m]} \times X^{[n]}$ and $\iota_{03} : X_0^{[\tilde{n}+m]} \times X_0^{[\tilde{n}]} \to X^{[n+m]} \times X_0^{[\tilde{n}]}$ by adding $\xi_1^{\lambda^1}$ suitably. By (2.21) and the projection formula, we obtain

$$\begin{aligned} \mathfrak{p}_{-m}([\Sigma_0])[\xi_\lambda] &= p_1^!\big(p_2^*[\xi_\lambda] \cup [(\Sigma_0)_{n,m}]\big) = p_1^!\big([X^{[n+m]} \times \xi_\lambda] \cup [(\Sigma_0)_{n,m}]\big) \\ &= p_1^!\big((\iota_{02})^![X^{[n+m]} \times \xi_0^{\lambda^0}] \cup [(\Sigma_0)_{n,m}]\big) \\ &= p_1^!(\iota_{02})^!\big([X^{[n+m]} \times \xi_0^{\lambda^0}] \cup (\iota_{02})^*[(\Sigma_0)_{n,m}]\big) \\ &= p_1^!(\iota_{02})^!\big([X^{[n+m]} \times \xi_0^{\lambda^0}] \cup (\iota_{03})^![(\Sigma_0)_{\tilde{n},m}]\big) \\ &= p_1^!(\iota_{02})^!(\iota_{03})^!\big((\iota_{03})^*[X^{[n+m]} \times \xi_0^{\lambda^0}] \cup [(\Sigma_0)_{\tilde{n},m}]\big) \\ &= p_1^!(\iota_{02})^!(\iota_{03})^!\big([X_0^{[\tilde{n}+m]} \times \xi_0^{\lambda^0}] \cup [(\Sigma_0)_{\tilde{n},m}]\big) \end{aligned}$$

where p_1 and p_2 are the projections of $X^{[n+m]} \times X^{[n]}$ to the two factors, and $(\Sigma_i)_{\tilde{n},m} \subset X_0^{[\tilde{n}+m]} \times X_0^{[\tilde{n}]}$ is defined similarly as in (2.20). Let $\tilde{p}_1$ and $\tilde{p}_2$ be the projections of $X_0^{[\tilde{n}+m]} \times X_0^{[\tilde{n}]}$ to the two factors. Then

$$\begin{aligned}
\mathfrak{p}_{-m}([\Sigma_0])[\xi_\lambda] &= p_1^!(\iota_{02})^!(\iota_{03})^!\big((\tilde{p}_2)^*[\xi_{\lambda^0}] \cup [(\Sigma_0)_{\tilde{n},m}]\big) \\
&= (\iota_{01})^!(\tilde{p}_1)^!\big((\tilde{p}_2)^*[\xi_{\lambda^0}] \cup [(\Sigma_0)_{\tilde{n},m}]\big) \\
&= (\iota_{01})^!\ \mathfrak{p}_{-m}([\Sigma_0])[\xi_{\lambda^0}].
\end{aligned} \tag{3.13}$$

Combining this with (3.12), we conclude that

$$\mathfrak{p}_{-m}([\Sigma_0])[\lambda] = -\sum_{|\mu|=m+|\lambda^0|} c(m,\lambda^0;\mu)[(\mu,\lambda^1)]. \tag{3.14}$$

Similarly, we can show that

$$\mathfrak{p}_{-m}([\Sigma_2])[\lambda] = \sum_{|\mu|=m+|\lambda^1|} c(m,\lambda^1;\mu)[(\lambda^0,\mu)]. \tag{3.15}$$

Recall that $[\Sigma_0]+[\Sigma_1]+[\Sigma_2]=0$ since $r=2$. So we have

$$\mathfrak{p}_{-m}([\Sigma_1])[\lambda] = \sum_{|\mu|=m+|\lambda^0|} c(m,\lambda^0;\mu)[(\mu,\lambda^1)] - \sum_{|\mu|=m+|\lambda^1|} c(m,\lambda^1;\mu)[(\lambda^0,\mu)]. \tag{3.16}$$

Thus Lemma 3.3 holds for $r=2$ by comparing with (3.6). Again, we stress that (3.14), (3.15) and (3.16) are still valid if the T-action on X is modified so that T acts on X_i $(i=0,1)$ by $s(z_{i,1},z_{i,2}) = (s^k z_{i,1}, s^{-k} z_{i,2})$ for a $k>0$.

Now assume $r>2$. Let $\lambda = (\lambda^0,\ldots,\lambda^{r-1}) \in \mathcal{P}_n(r)$. The same arguments as in the proof of (3.14) show that Lemma 3.3 is true for $i=0$ or r. Let $1 \le i \le (r-1)$. Then the T-fixed points in the projective line Σ_i are ξ_{i-1} and ξ_i. Let $\widetilde{X} = X_{i-1} \cup X_i$ be equipped with the induced T-action. Then, ξ_{i-1} and ξ_i are the only T-fixed points in $\widetilde{X}$, and we can apply the results in the previous paragraph to $\widetilde{X}$. The T-equivariant inclusion map $\tilde{\iota} : \widetilde{X} \to X$ induces the T-equivariant inclusion map $\tilde{\iota}_1 : \widetilde{X}^{[\tilde{n}+m]} \to X^{[n+m]}$ which sends ξ to $\xi + \sum_{0\le j\le r-1,\, j\ne i-1,i} \xi_j^{\lambda^j}$, where $\tilde{n} = |\lambda^{i-1}|+|\lambda^i|$. Put $\tilde{\lambda} = (\lambda^{i-1},\lambda^i) \in \mathcal{P}_{\tilde{n}}(2)$ and $\xi_{\tilde{\lambda}} = \xi_{i-1}^{\lambda^{i-1}} + \xi_i^{\lambda^i} \in \widetilde{X}^{[\tilde{n}]}$. Arguments parallel to the proof of (3.13) show that

$$\mathfrak{p}_{-m}([\Sigma_i])[\xi_\lambda] = (\tilde{\iota}_1)^!\ \mathfrak{p}_{-m}([\Sigma_i])[\xi_{\tilde{\lambda}}].$$

Combining this with (3.16), we conclude that

$$\begin{aligned}
\mathfrak{p}_{-m}([\Sigma_i])[\lambda] \quad = &\sum_{|\mu|=m+|\lambda^{i-1}|} c(m,\lambda^{i-1};\mu)[(\lambda^0,\ldots,\lambda^{i-2},\mu,\lambda^i,\lambda^{i+1},\ldots,\lambda^{r-1})] \\
&- \sum_{|\mu|=m+|\lambda^i|} c(m,\lambda^i;\mu)[(\lambda^0,\ldots,\lambda^{i-2},\lambda^{i-1},\mu,\lambda^{i+1},\ldots,\lambda^{r-1})].
\end{aligned}$$

Now we complete the proof of Lemma 3.3 for $r>2$ by comparing with (3.6). $\square$

THEOREM 3.4. *There is a canonical ring isomorphism $\phi : \mathbb{H}_n \longrightarrow R(\Gamma_n)$ satisfying*

$$\begin{aligned}
\phi([\lambda]) &= s_\lambda \\
\phi(\mathfrak{p}_{-\lambda}) &= \mathfrak{a}_{-\lambda}^R \qquad (3.17)\\
\phi(\mathfrak{p}'_{-\lambda}) &= \mathfrak{a}_{-\lambda} \qquad (3.18)
\end{aligned}$$

for each $\lambda \in \mathcal{P}_n(r)$. In addition, ϕ is an isometry with respect to the standard bilinear forms on $\mathbb{H}$ and $\mathbb{R}_\Gamma$, and commutes with the action of Heisenberg algebra.

PROOF. We have noted that the map ϕ defined by (3.11) is a linear isomorphism. The map ϕ is an isometry since for two arbitrary λ and μ in $\mathcal{P}_n(r)$, we have

$$\langle [\lambda], [\mu] \rangle = \delta_{\lambda,\mu} = \langle s_\lambda, s_\mu \rangle.$$

The compatibility of ϕ with the Heisenberg creation operators was verified in Lemma 3.3. Since the annihilation operators are the adjoints of the creation operators with respect to the bilinear forms, they are also compatible with ϕ.

By (2.14), (2.17) and the definition of $\star$, we obtain

$$[\lambda] \star [\mu] = \delta_{\lambda,\mu} r^n h(\lambda)\ [\lambda].$$

On the other hand, it follows from (3.2) and (3.7) that

$$s_\lambda \circ s_\mu = \delta_{\lambda,\mu} r^n h(\lambda)\ s_\lambda.$$

Thus $\phi : \mathbb{H}_n \longrightarrow R(\Gamma_n)$ is actually a ring isomorphism.

Lemma 3.3 implies that for each $m > 0$ and $\lambda \in \mathcal{P}_n(r)$, we have

$$\phi\big(\mathfrak{p}_{-m}(\diamondsuit_i) \cdot [\lambda]\big) = \mathfrak{a}_{-m}(\gamma_i) \cdot s_\lambda, \qquad 0 \le i \le r-1. \tag{3.19}$$

Now (3.17) follows from an induction argument and the definitions of $\mathfrak{p}_{-\lambda}$ and $\mathfrak{a}^R_{-\lambda}$. Finally, note from definitions that the transition matrix between $\mathfrak{p}_{-\lambda}$ and $\mathfrak{p}'_{-\lambda}$ coincides with the one between $\mathfrak{a}^R_{-\lambda}$ and $\mathfrak{a}_{-\lambda}$. Thus (3.18) follows from (3.17). $\square$

3.4. An isomorphism for ordinary cohomology rings. By Lemma 2.1 (i), we have $rt = \sum_{i=0}^{r-1} \diamondsuit_i$. On the other hand, $c^0 = \sum_{i=0}^{r-1} \gamma_i$. It follows from (3.19) that, for each $m > 0$ and $\lambda \in \mathcal{P}_n(r)$,

$$\phi\big(\mathfrak{p}_{-m}(rt) \cdot [\lambda]\big) = \mathfrak{a}_{-m}(c^0) \cdot s_\lambda. \tag{3.20}$$

Denoting $\lambda = (\lambda^0, \cdots, \lambda^{r-1})$ and $\lambda^i = (1^{m_1(i)} 2^{m_2(i)} \cdots)$, we define

$$\begin{aligned} \mathfrak{b}_{-\lambda} &= \prod_{k \ge 1} \left(\mathfrak{a}_{-k}(c^0)^{m_k(0)} \prod_{i=1}^{r-1} \mathfrak{a}_{-k}(\gamma_{i-1} - \gamma_i)^{m_k(i)} \right) |0\rangle \in \mathbb{R}_\Gamma \\ \mathfrak{q}^T_{-\lambda} &= \prod_{k \ge 1} \left(\mathfrak{p}_{-k}(rt)^{m_k(0)} \prod_{i=1}^{r-1} \mathfrak{p}_{-k}([\Sigma_i])^{m_k(i)} \right) |0\rangle \in \mathbb{H}. \end{aligned} \tag{3.21}$$

It follows by induction from Lemma 3.3 and (3.20) that

$$\phi(\mathfrak{q}^T_{-\lambda}) = \mathfrak{b}_{-\lambda}. \tag{3.22}$$

Note that a linear basis for the ordinary cohomology $H^*(X)$ is given by

$$1_X \in H^0(X), \qquad \Sigma_1, \ldots, \Sigma_{r-1} \in H^2(X).$$

Using the usual Heisenberg algebra construction [**Na1**] (in Nakajima's notations, the Heisenberg operators were denoted by $P_{1_X}[m]$ and $P_{\Sigma_i}[m]$), we can construct a linear basis $\{\mathfrak{q}_{-\lambda}\}_{\lambda \in \mathcal{P}_n(r)}$ for $H^*(X^{[n]})$, where we have denoted

$$\mathfrak{q}_{-\lambda} = \prod_{k \ge 1} \left(P_{r \cdot 1_X}[-k]^{m_k(0)} \prod_{i=1}^{r-1} P_{\Sigma_i}[-k]^{m_k(i)} \right) |0\rangle \tag{3.23}$$

if $\lambda = (\lambda^0, \dots, \lambda^{r-1})$ with $\lambda^i = (1^{m_1(i)} 2^{m_2(i)} \cdots)$. Using the degrees of the Heisenberg operators (cf. [**Na2**]), the cohomology degree of $\mathfrak{q}_{-\lambda}$ is computed to be

$$\sum_{k \geq 1} \left(2(k-1)m_k(0) + \sum_{i=1}^{r-1} (2(k-1)+2)m_k(i) \right) = 2\big(\|\lambda\| - \ell(\lambda^0)\big).$$

Let $F^p = \sum_{k \leq p} t^{n-k} H_T^{2k}(X^{[n]})$. Then

$$F^0 \subset F^1 \subset \ldots \subset F^n = \mathbb{H}_n$$

defines a filtration on the ring $\mathbb{H}_n$. By a spectral sequence argument (cf. [**Na3, Vas**]), $H^*(X^{[n]})$ is identified with the graded ring of $\mathbb{H}_n$ associated to this filtration, and $\mathfrak{q}_{-\lambda} \in H^*(X^{[n]})$ is the element associated to $\mathfrak{q}^T_{-\lambda} \in \mathbb{H}_n$.

On the other hand, note that $(\gamma_j - \gamma_{j+1})$ is homogeneous in $\mathcal{G}_\Gamma(1)$ of degree 1 and c^0 is of degree 0. By the definition of the degree of a conjugacy class of Γ_n in Sect. 3.2 and Remark 3.1, the degree of $\mathfrak{b}_{-\lambda}$ equals $\|\lambda\| - \ell(\lambda^0)$.

Thus, by (3.22) above, the filtration on the ring $\mathbb{H}_n$ is compatible (up to a multiple 2) with the degree filtration on $R(\Gamma_n)$ defined in Sect. 3.2. We can regard $\mathfrak{b}_{-\lambda}$ as an element in $\mathcal{G}^*_\Gamma(n)$. Further note that c^0 is r times the identity element of the ring $\mathcal{G}^*_\Gamma(1)$, which is compatible with $r \cdot 1^X$ in $H^0(X)$. Also compare (3.21) and (3.23). Summarizing, we have established the following.

THEOREM 3.5. *Rescale the grading on $\mathcal{G}^*_\Gamma(n)$ by a multiple of 2. Then, the map from $H^*(X^{[n]})$ to $\mathcal{G}^*_\Gamma(n)$ which sends $\mathfrak{q}_{-\lambda}$ to $\mathfrak{b}_{-\lambda}$ for each $\lambda \in \mathcal{P}_n(r)$ is a graded ring isomorphism.* □

REMARK 3.6. When $r = 1$, i.e. Γ is trivial, X is the affine plane and Γ_n is the symmetric group S_n. In this case, our Theorem 3.4 specializes to a theorem in [**Vas**], and Theorem 3.5 specializes to a theorem in [**LS, Vas**] (also cf. [**LQW1**]). Theorem 3.5 further supports and enhances Ruan's conjecture on the existence of an isomorphism between the cohomology ring of a hyperkahler resolution and Chen-Ruan orbifold cohomology ring, [**Ru**] (also cf. [**Wa3**]). Ruan's conjecture does not provide any explicit map which realizes a ring isomorphism. As a special case of a more general theorem in [**EG**], the ring isomorphism in Theorem 3.5 without the explicit map has been verified in a very indirect way.

REMARK 3.7. The structure constants of the algebra $\mathcal{G}^*_\Gamma(n)$ in the basis of conjugacy classes of Γ_n are shown to be independent of n, and in addition, they are non-negative integers [**Wa3**]. In a completely different approach [**LQW1**], the structure constants of the cup product of the Heisenberg monomial basis for $H^*(X^{[n]})$ are shown to be independent of n. In light of Theorem 3.5, these two "independent of n" statements are equivalent. Nevertheless, the positivity and integrality of the structure constants in $R(\Gamma_n)$, when transferred over to $\mathbb{H}_n$, are still waiting for a geometric interpretation (even when Γ is trivial).

4. Moduli spaces of sheaves and fermionic Fock space

4.1. The moduli spaces $\mathcal{M}(p, n)$ of sheaves. Denote by P the rank-r integral lattice in $\mathbb{H}_1 = H^2_T(X)$ which is $\mathbb{Z}$-spanned by $\diamondsuit_i, 0 \leq i \leq r-1$, with the inherited bilinear form such that $\langle \diamondsuit_i, \diamondsuit_j \rangle = \delta_{ij}$. We denote by $\mathcal{O}_m$, $m \in \mathbb{Z}$, the line bundle $X \times_T \mathbb{C}$ over X, where T acts on $\mathbb{C}$ according to the representation θ^m.

LEMMA 4.1. *Given $p \in P$, there exists a T-equivariant line bundle over X whose T-equivariant first Chern class is p.*

PROOF. It suffices to show that there exists a T-equivariant line bundle over X whose T-equivariant first Chern class is $\diamondsuit_i, 0 \le i \le r-1$. By (2.33),

$$c_1^T(L_k) = \frac{1}{r}\left(\sum_{i=0}^{k-1}(k-r)\diamondsuit_i + \sum_{i=k}^{r-1} k\diamondsuit_i\right), \qquad 0 \le k \le r-1. \tag{4.1}$$

Combined with Lemma 2.1, this implies that

$$\begin{aligned} \diamondsuit_k &= c_1^T(L_k) - c_1^T(L_{k+1}) + t, \qquad 0 \le k < r-1 \\ \diamondsuit_{r-1} &= c_1^T(L_{r-1}) + t. \end{aligned} \tag{4.2}$$

Note that the T-equivariant first Chern class of the line bundle $\mathcal{O}_m$ equals mt. Therefore, for $0 \le k < r-1$, the T-equivariant first Chern class of $L_k \otimes L_{k+1}^\vee \otimes \mathcal{O}_1$ is $\diamondsuit_k$, and the T-equivariant first Chern class of $L_{r-1} \otimes \mathcal{O}_1$ is $\diamondsuit_{r-1}$. □

REMARK 4.2. We derive from Lemma 2.1 and (4.2) that

$$([\Sigma_1], [\Sigma_2], \cdots, [\Sigma_{r-1}]) = -(c_1^T(L_1), c_1^T(L_2), \cdots, c_1^T(L_{r-1})) \cdot C_{r-1}$$

where C_{r-1} is the Cartan matrix of type A_{r-1}. This can be regarded as a equivariant geometric version of the McKay correspondence [**McK**]. A relation of this sort in the setup of ordinary cohomology theory of the minimal resolutions was due to Gonzalez-Sprinberg and Verdier [**GSV**].

Denote by $\mathcal{O}(p)$ the T-equivariant line bundle (guaranteed by the above Lemma) over X whose T-equivariant first Chern class is $p \in P$. We introduce the moduli space, denoted by $\mathcal{M}(p,n)$, which parameterizes all rank-1 subsheaves of $\mathcal{O}(p)$ such that the quotients are supported at finitely many points of X and have length n. Given $\mathcal{I} \in X^{[n]}$, then $\mathcal{O}(p) \otimes \mathcal{I}$ is an element in $\mathcal{M}(p,n)$. Therefore, these moduli spaces for a fixed n and varied p are all T-equivariantly isomorphic to $X^{[n]}$. In particular, $\mathcal{M}(0,n) = X^{[n]}$. As before, the consideration of the equivariant cohomology ring $H_T^*(\mathcal{M}(p,n))$ leads to a ring $\mathbb{H}_n^{(p)} \stackrel{\text{def}}{=} H_T^{2n}(\mathcal{M}(p,n))$ whose product is again denoted by $\star$. For $p \in P$, we introduce

$$\mathbb{H}^{(p)} = \bigoplus_{n=0}^{\infty} \mathbb{H}_n^{(p)}, \qquad \mathcal{F} = \bigoplus_{p \in P} \mathbb{H}^{(p)}.$$

The natural identification $\mathcal{M}(p,n) \cong X^{[n]}$ leads to the natural identification of the rings $\mathbb{H}_n^{(p)} \cong \mathbb{H}_n$, which induces a bilinear form $\langle -, - \rangle^{(p)}$ on $\mathbb{H}_n^{(p)}$ and $\mathbb{H}^{(p)}$. Similarly, we shall add the superscript (p) to denote the counterparts in $\mathcal{M}(p,n)$ of the objects in $X^{[n]}$ and its equivariant cohomology, such as $\xi_\lambda^{(p)}$, $[\lambda]^{(p)}$, etc.

Given $\alpha \in P$, we denote by S_α the isomorphism from $\mathbb{H}_n^{(p)}$ to $\mathbb{H}_n^{(p+\alpha)}$. This induces isomorphisms $S_\alpha : \mathcal{F} \to \mathcal{F}$ and $S_\alpha : \mathbb{H}^{(p)} \to \mathbb{H}^{(p+\alpha)}$ for all $p \in P$. Apparently, we have $S_{\alpha+\beta} = S_\alpha S_\beta$ for $\alpha, \beta \in P$.

4.2. The Chern character operators on fermionic Fock space. Let $p \in P$. Denote by π_1 and π_2 the projections of $\mathcal{M}(p,n) \times X$ to the two factors. Over $\mathcal{M}(p,n) \times X$, we have a universal exact sequence:

$$0 \longrightarrow \mathcal{J}(p) \longrightarrow \pi_2^* \, \mathcal{O}(p) \longrightarrow \mathcal{Q}(p) \to 0.$$

For $0 \le k \le r-1$, we denote by $L_k(p)^{[n]}$ the T-equivariant rank n vector bundle over $\mathcal{M}(p,n)$ given by the pushforward $\pi_{1*}(\mathcal{Q}(p)\otimes \pi_2^* L_k)$, whose fiber over a fixed point $\xi_\lambda^{(p)} \in \mathcal{M}(p,n) \cong X^{[n]}$ is given by

$$L_k(p)^{[n]}\,|_{\xi_\lambda^{(p)}} = \mathcal{O}(p)\otimes L_k^{[n]}\,|_{\xi_\lambda^{(p)}}, \qquad \lambda \in \mathcal{P}_n(r). \tag{4.3}$$

In the same way as defining the operators $\mathfrak{G}_k(z)$ and $\mathfrak{G}_{k;m}$ which act on $\mathbb{H} = \mathbb{H}^{(0)}$, where $0 \le k \le r-1$ and $m \ge 0$, we can define the operators $\mathfrak{G}_k^{(p)}(z)$ (respectively, $\mathfrak{G}_{k;m}^{(p)}$) acting on $\mathbb{H}^{(p)}$ by the $\star$-product with $\sum_{m\ge 0} t^{n-m}\mathrm{ch}_m^T(L_k(p)^{[n]})\, z^m$ (respectively, with $t^{n-m}\mathrm{ch}_m^T(L_k(p)^{[n]})$) on $\mathbb{H}_n^{(p)}$ for each n.

Take $p = \sum_i n_i \diamondsuit_i \in P$, where $n_i \in \mathbb{Z}$. One can show that as T-modules

$$\mathcal{O}(\diamondsuit_i)|_{\xi_j} \cong \theta^{r\delta_{i,j}}.$$

Similarly as in Subsection 2.4, by the projection formula and (4.3), we obtain

$$\begin{aligned}\mathrm{ch}_m^T(L_k(p)^{[n]}) \cup [\xi_\lambda^{(p)}] = \frac{1}{m!}\Bigg(&\sum_{i=0}^{k-1}\sum_{\square\in\lambda^i}\big((n_i r + k - r + r c_\square)t\big)^m \\ &+ \sum_{i=k}^{r-1}\sum_{\square\in\lambda^i}\big((n_i r + k + r c_\square)t\big)^m\Bigg)[\xi_\lambda^{(p)}].\end{aligned} \tag{4.4}$$

Lemma 4.3. *Let $p = \sum_i n_i\diamondsuit_i \in P$, where $n_i \in \mathbb{Z}$, and let $\lambda = (\lambda^0, \cdots, \lambda^{r-1}) \in \mathcal{P}_n(r)$ with $\lambda^i = (\lambda_1^i, \lambda_2^i, \ldots)$. Then, for $0 \le k \le r-1$, we have*

$$\begin{aligned}\mathfrak{G}_k^{(p)}(z)\cdot[\lambda]^{(p)} \;=\; \frac{1}{\varsigma(rz)}\Bigg(&\sum_{i=0}^{k-1} e^{(n_i r + k - r)z}\left(\sum_{j=1}^{\infty} e^{(\lambda_j^i - j + 1/2)rz} - \frac{1}{\varsigma(rz)}\right) \\ &+ \sum_{i=k}^{r-1} e^{(n_i r + k)z}\left(\sum_{j=1}^{\infty} e^{(\lambda_j^i - j+1/2)rz} - \frac{1}{\varsigma(rz)}\right)\Bigg)\;[\lambda]^{(p)}.\end{aligned}$$

Proof. Follows from (2.36) and (4.4) directly. $\square$

For later purpose, we introduce the following modified classes

$$\mathrm{ch}_{k;m}^{[n](p)} = t^{n-m}\mathrm{ch}_m^T(L_k(p)^{[n]}) + c_{k;m}^{(p)} t^n, \qquad m \ge -1 \tag{4.5}$$

where the constant $c_{k;m}^{(p)}$ is defined by

$$\sum_{m\ge -1} c_{k;m}^{(p)} z^m = \frac{1}{\varsigma(rz)^2}\left(e^{(k-r)z}\sum_{i=0}^{k-1}(e^{n_i rz}-1) + e^{kz}\sum_{i=k}^{r-1}(e^{n_i rz}-1)\right)$$

and $\mathrm{ch}_{-1}^T(L_k(p)^{[n]}) = 0$ by convention. Equivalently, if we define

$$\begin{aligned}\widetilde{\mathfrak{G}}_k^{(p)}(z) =\, &\mathfrak{G}_k^{(p)}(z) + \\ &\frac{1}{\varsigma(rz)^2}\left(e^{(k-r)z}\sum_{i=0}^{k-1}(e^{n_i rz}-1) + e^{kz}\sum_{i=k}^{r-1}(e^{n_i rz}-1)\right)\mathrm{Id}\end{aligned} \tag{4.6}$$

and further write

$$\widetilde{\mathfrak{G}}_k^{(p)}(z) = \sum_{m=-1}^{\infty} \widetilde{\mathfrak{G}}_{k;m}^{(p)} z^m,$$

then

$$\widetilde{\mathfrak{G}}^{(p)}_{k;m} = \mathfrak{G}^{(p)}_{k;m} + c^{(p)}_{k;m}\mathrm{Id},$$

(where we take the convention that $\mathfrak{G}^{(p)}_{k;-1} = 0$), and $\widetilde{\mathfrak{G}}^{(p)}_{k;m}$ acts on $\mathbb{H}^{(p)}_n$ by the product with $\mathrm{ch}^{[n](p)}_{k;m}$. When $p = 0$, we have

$$c^{(0)}_{k;m} = 0, \quad \mathrm{ch}^{[n](0)}_{k;m} = \mathrm{ch}^{[n]}_{k;m}, \quad \widetilde{\mathfrak{G}}^{(0)}_{k;m} = \mathfrak{G}_{k;m}, \quad \widetilde{\mathfrak{G}}^{(0)}_{k}(z) = \mathfrak{G}_{k}(z).$$

By the standard boson-fermion correspondence (cf. [**MJD**]), we shall identify $\mathcal{F}$ with the fermionic Fock space of r pairs of fermions $\psi^{+,k}(z), \psi^{-,k}(z), 0 \le k \le r-1$. More explicitly, we extend the Heisenberg operators $\mathfrak{p}_n(\alpha)$, where $\alpha \in \mathbb{H}_1$ and $n \neq 0$, to act on $\mathbb{H}^{(p)}, p \in P$ via the identification $\mathbb{H}^{(p)} \cong \mathbb{H}$, and let $\mathfrak{p}_0(\alpha)$ act on $\mathbb{H}^{(p)}$ by $\langle \alpha, p\rangle\,\mathrm{Id}$. Note that this is compatible with $\mathfrak{p}_0(\alpha) = 0$ on $\mathbb{H}$ thanks to $\mathbb{H} = \mathbb{H}^{(0)}$. The fermionic fields can be expressed as

$$\psi^{\pm,k}(z) = S_{\diamondsuit_k} z^{\pm \mathfrak{p}_0(\diamondsuit_k)} \exp\left(\sum_{n<0} \mp \frac{\mathfrak{p}_n(\diamondsuit_k)}{n} z^{-n}\right) \exp\left(\sum_{n>0} \mp \frac{\mathfrak{p}_n(\diamondsuit_k)}{n} z^{-n}\right).$$

It has been well known (cf. e.g. [**MJD**]) that the completed infinite-rank general linear Lie algebra $\widehat{gl}_\infty$, whose standard basis is denoted by $E_{i,j},\ i,j \in \mathbb{Z} + 1/2$, acts on the fermionic Fock space of a pair of fermions. Thus, we have an *orthogonal* direct sum of r copies of $\widehat{gl}_\infty$, denoted by $\widehat{gl}^{\,r}_\infty$, acting on $\mathcal{F}$, whose generators will be denoted by $E^{(k)}_{i,j}, 0 \le k \le r-1$. Using the generating field

$$E^{(k)}(z,w) = \sum_{i,j \in \mathbb{Z}+1/2} E^{(k)}_{i,j} z^{i-1/2} w^{-j-1/2}$$

the action of $\widehat{gl}^{\,r}_\infty$ is simply given by

$$E^{(k)}(z,w) = :\psi^{+,k}(z)\psi^{-,k}(w):.$$

We refer to *loc. cit.* for the standard notation of normal ordering : :. Denote

$$\mathcal{E}^{(i)}(z) = \frac{1}{\varsigma(rz)} \sum_{m \in \mathbb{Z}+\frac{1}{2}} e^{mrz} E^{(i)}_{m,m}.$$

We introduce the following operator in $\mathrm{End}(\mathcal{F})$:

$$\mathfrak{H}_k(z) = e^{(k-r)z} \sum_{i=0}^{k-1} \mathcal{E}^{(i)}(z) + e^{kz} \sum_{i=k}^{r-1} \mathcal{E}^{(i)}(z) \tag{4.7}$$

which is further expanded as

$$\mathfrak{H}_k(z) = \sum_{m=-1}^{\infty} \mathfrak{H}_{k;m} z^m, \qquad 0 \le k \le r-1.$$

Alternatively, we can rewrite (4.7) as

$$\mathfrak{H}_k(z) = \sum_{i=0}^{r-1} e^{a(k,i)z} \mathcal{E}^{(i)}(z) \tag{4.8}$$

where

$$a(k,i) = \begin{cases} k-r, & \text{if } 0 \le i \le k-1 \\ k, & \text{if } k \le i \le r-1 \end{cases} \tag{4.9}$$

For example, $\mathfrak{H}_{k;-1} = \sum_{i=0}^{r-1} r^{-1}\mathfrak{p}_0(\diamondsuit_i)$. It is well known (cf. [**MJD**]) that the elements in the Cartan subalgebra of $\widehat{gl}_\infty$ diagonalize the basis of Schur functions. Thanks to the identification of the $[\lambda]$'s with multi-variable Schur functions in Theorem 3.4, we have the following straightforward multi-variable generalization in our context (cf. Lemmas 3.1 and 3.6 in [**LQW2**]; also cf. [**OP**]).

LEMMA 4.4. *The operators $\mathcal{E}^{(i)}(z)$ and $\mathfrak{H}_k(z)$ diagonalize the elements $[\lambda]$, $\lambda \in \cup_n \mathcal{P}_n(r)$. More explicitly, for $\lambda = (\lambda^0, \cdots, \lambda^{r-1})$ with $\lambda^i = (\lambda^i_1, \lambda^i_2, \cdots)$, we have*

$$\mathcal{E}^{(i)}(z) \cdot [\lambda] = \frac{1}{\varsigma(rz)} \left(\sum_{j=1}^{\infty} e^{(\lambda^i_j - j + 1/2)rz} - \frac{1}{\varsigma(rz)} \right) [\lambda]. \qquad \square$$

THEOREM 4.5. *Given $p \in P$, we have the identification*

$$\mathfrak{H}_k(z) \mid_{\mathbb{H}^{(p)}} = \widetilde{\mathfrak{G}}^{(p)}_k(z).$$

PROOF. First consider $p = 0$. It follows from Proposition 2.3, the definition (4.7) of $\mathfrak{H}_k(z)$, and Lemma 4.4 that $\mathfrak{H}^{(k)}(z)$ and $\mathfrak{G}_k(z)$ have the same eigenvalues on the basis elements $[\lambda]$ of $\mathbb{H}^{(0)}$, whence

$$\mathfrak{H}^{(k)}(z) \mid_{\mathbb{H}^{(0)}} = \mathfrak{G}_k(z) = \widetilde{\mathfrak{G}}^{(0)}_k(z).$$

Now let $p = \sum_i n_i \diamondsuit_i$, $n_i \in \mathbb{Z}$. For $d \in \mathbb{Z}$, we can show by a standard computation that (cf. Lemma 3.5, [**LQW2**])

$$S^{-d}_{\diamondsuit_i}\, \mathcal{E}^{(j)}(z)\, S^d_{\diamondsuit_i} \quad = \quad \begin{cases} e^{drz}\mathcal{E}^{(i)}(z) + \frac{e^{drz}-1}{\varsigma(rz)^2}\,\mathrm{Id}, & \text{if } i = j \\ \mathcal{E}^{(j)}(z), & \text{if } i \neq j \end{cases}$$

Thus, using (4.7) we have

$$\begin{aligned} S_p^{-1}\, \mathfrak{H}_k(z)\, S_p &= \prod_i S^{-n_i}_{\diamondsuit_i}\, \mathfrak{H}_k(z) \prod_i S^{n_i}_{\diamondsuit_i} \\ &= e^{(k-r)z} \sum_{i=0}^{k-1} e^{n_i rz}\mathcal{E}^{(i)}(z) + e^{kz} \sum_{i=k}^{r-1} e^{n_i rz}\mathcal{E}^{(i)}(z) \\ &\quad + \left(\sum_{i=0}^{k-1} \frac{e^{(k-r)z}(e^{n_i rz} - 1)}{\varsigma(rz)^2} + \sum_{i=k}^{r-1} \frac{e^{kz}(e^{n_i rz} - 1)}{\varsigma(rz)^2} \right) \mathrm{Id}. \end{aligned}$$

Together with Lemma 4.4, this implies by a little algebra manipulation that

$$\begin{aligned} \mathfrak{H}_k(z) \cdot [\lambda]^{(p)} &= \mathfrak{H}_k(z) S_p \cdot [\lambda] \\ &= \frac{1}{\varsigma(rz)} \left(\sum_{i=0}^{k-1} \left(\sum_{j=1}^{\infty} e^{(n_i r + k - r)z} e^{(\lambda^i_j - j + 1/2)rz} - \frac{e^{(k-r)z}}{\varsigma(rz)} \right) \right. \\ &\qquad \left. + \sum_{i=k}^{r-1} \left(\sum_{j=1}^{\infty} e^{(n_i r + k)z} e^{(\lambda^i_j - j + 1/2)rz} - \frac{e^{kz}}{\varsigma(rz)} \right) \right) [\lambda]^{(p)}. \end{aligned}$$

On the other hand, by (4.6) and Lemma 4.3, we have

$$\begin{aligned}\widetilde{\mathfrak{G}}^{(p)}_k(z)\cdot[\lambda]^{(p)} &= \frac{1}{\varsigma(rz)}\left(\sum_{i=0}^{k-1}\left(\sum_{j=1}^{\infty}e^{(n_ir+k-r)z}e^{(\lambda^i_j-j+1/2)rz}-\frac{e^{(k-r)z}}{\varsigma(rz)}\right)\right.\\ &\quad \left.+\sum_{i=k}^{r-1}\left(\sum_{j=1}^{\infty}e^{(n_ir+k)z}e^{(\lambda^i_j-j+1/2)rz}-\frac{e^{kz}}{\varsigma(rz)}\right)\right)\ [\lambda]^{(p)}.\end{aligned}$$

Thus, we see that the two operators $\mathfrak{H}_k(z)$ and $\widetilde{\mathfrak{G}}^{(p)}_k(z)$ have the same eigenvalues on every $[\lambda]^{(p)}$. This proves the theorem. $\square$

5. Generating functions of intersection numbers

5.1. The N-point functions of intersection numbers. We are interested in the equivariant intersection numbers for $X^{[n]}$ (associated to $\lambda,\mu\in\mathcal{P}_n(r)$):

$$\left\langle \lambda, \mathrm{ch}^{[n]}_{k_1;m_1}\cdots\mathrm{ch}^{[n]}_{k_N;m_N},\mu\right\rangle := \left\langle \mathfrak{p}_{-\lambda}, \mathrm{ch}^{[n]}_{k_1;m_1}\star\cdots\star\mathrm{ch}^{[n]}_{k_N;m_N}\star\mathfrak{p}_{-\mu}\right\rangle$$

where the classes $\mathrm{ch}^{[n]}_{k_i;m_i}$ are defined in (2.34). We can organize these intersection numbers into the *N-point function* (for fixed $0\le k_1,\ldots,k_N\le r-1$):

$$\begin{aligned}& G_{\lambda,\mu;r}(z_1,\ldots,z_N;k_1,\ldots,k_N)\\ =& \sum_{m_1,\ldots,m_N=0}^{\infty} z_1^{m_1}\cdots z_N^{m_N}\left\langle \lambda, \mathrm{ch}^{[n]}_{k_1;m_1}\cdots\mathrm{ch}^{[n]}_{k_N;m_N},\mu\right\rangle,\end{aligned}$$

which can be further reformulated in an operator form thanks to Theorem 4.5:

$$\begin{aligned}G_{\lambda,\mu;r}(z_1,\ldots,z_N;k_1,\ldots,k_N) &= \langle\mathfrak{p}_{-\lambda},\mathfrak{G}_{k_1}(z_1)\cdots\mathfrak{G}_{k_N}(z_N)\mathfrak{p}_{-\mu}\rangle\\ &= \langle\mathfrak{p}_{-\lambda},\mathfrak{H}_{k_1}(z_1)\cdots\mathfrak{H}_{k_N}(z_N)\mathfrak{p}_{-\mu}\rangle.\end{aligned}\tag{5.1}$$

For latter purpose, we have indicated r explicitly in the notation above. When $r=1$ (i.e. $X=\mathbb{C}^2$), we simply write it as $G_{\lambda,\mu;1}(z_1,\ldots,z_N)$.

We shall need the notion of a partition of the set $\{1,\ldots,N\}$ which is a collection of subsets $\mathcal{N}:=(\mathcal{N}_0,\ldots,\mathcal{N}_{r-1})$ whose disjoint union is $\{1,\ldots,N\}$. Denote $\mathbf{z}=(z_1,\cdots,z_N)$ and $\mathbf{k}=(k_1,\cdots,k_N)$, and let

$$F(\mathbf{z},\mathbf{k};\mathcal{N}) := \prod_{i=0}^{r-1}\prod_{j\in\mathcal{N}_i} e^{a(k_j,i)z_j}.$$

(See (4.9) for notations). The next proposition reduces the calculation of the N-point function for a general r to the case $r=1$.

Proposition 5.1. *Fix $\lambda=(\lambda^0,\ldots,\lambda^{r-1})$ and $\mu=(\mu^0,\ldots,\mu^{r-1})$ in $\mathcal{P}_n(r)$. Then*

$$G_{\lambda,\mu;r}(z_1,\ldots,z_N;k_1,\ldots,k_N)=0$$

unless $|\lambda^i|=|\mu^i|$ for $0\le i\le r-1$; in this case, then

$$G_{\lambda,\mu;r}(z_1,\ldots,z_N;k_1,\ldots,k_N) = \sum_{\mathcal{N}}\left(F(\mathbf{z},\mathbf{k};\mathcal{N})\prod_{i=0}^{r-1}G_{\lambda^i,\mu^i;1}(rz_{l_{i,1}},\ldots,rz_{l_{i,n_i}})\right)$$

summed over all partitions $\mathcal{N}=(\mathcal{N}_0,\ldots,\mathcal{N}_{r-1})$ of the set $\{1,\ldots,N\}$, where $\mathcal{N}_i=\{l_{i,1},\ldots,l_{i,n_i}\}$.

PROOF. We can write $\mathfrak{a}^R_{-\mu}$ as a linear combination: $\mathfrak{a}^R_{-\mu} = \sum_{\nu\in\mathcal{P}_n(r)} d_{\mu,\nu} s_\nu$, where ν satisfies $|\nu^i| = |\mu^i|$ for $0 \le i \le r-1$, cf. Section 3.1. By Theorem 3.4, we have $\mathfrak{p}_{-\mu} = \sum_{\nu\in\mathcal{P}_n(r)} d_{\mu,\nu}[\nu]$. The same can be done for $\mathfrak{p}_{-\lambda}$. Since the $[\nu]$'s are orthogonal eigenvectors for the operator $\mathfrak{G}_k(z)$, we see that

$$G_{\lambda,\mu;r}(z_1,\ldots,z_N;k_1,\ldots,k_N) = 0$$

unless $|\lambda^i| = |\mu^i|$ for $0 \le i \le r-1$.

Now assume $|\lambda^i| = |\mu^i|$ for $0 \le i \le r-1$. Recall from (2.28) that

$$\mathfrak{p}_{-\mu} = \prod_i \mathfrak{p}_{-\mu^i}(\diamondsuit_i)|0\rangle.$$

Note that the operator $\mathcal{E}^{(j)}(z)$ commutes with the operators $\mathcal{E}^{(i)}(z)$ and $\mathfrak{p}_{-\mu^i}(\diamondsuit_i)$ for $j \ne i$. Thus, by (4.8) and (5.1), we obtain

$$\begin{aligned}
(5.2)\qquad & G_{\lambda,\mu;r}(z_1,\ldots,z_N;k_1,\ldots,k_N)\\
&= \sum_{\mathcal{N}} \left\langle \mathfrak{p}_{-\lambda}, \prod_{i=0}^{r-1}\prod_{j\in\mathcal{N}_i}\left(e^{a(k_j,i)z_j}\mathcal{E}^{(i)}(z_j)\right)\mathfrak{p}_{-\mu}\right\rangle\\
&= \sum_{\mathcal{N}} F(\mathbf{z},\mathbf{k};\mathcal{N})\cdot\left\langle \mathfrak{p}_{-\lambda}, \prod_{i=0}^{r-1}\prod_{j\in\mathcal{N}_i}\mathcal{E}^{(i)}(z_j)\mathfrak{p}_{-\mu^i}(\diamondsuit_i)|0\rangle\right\rangle
\end{aligned}$$

summed over all partitions $\mathcal{N} = (\mathcal{N}_0,\ldots,\mathcal{N}_{r-1})$ of the set $\{1,\ldots,N\}$.

Observe that we can write $\prod_{j\in\mathcal{N}_i}\mathcal{E}^{(i)}(z_j)\mathfrak{p}_{-\mu^i}(\diamondsuit_i)|0\rangle$ as a linear combination of $\mathfrak{p}_{-\nu^i}(\diamondsuit_i)|0\rangle$'s, and thus, $\prod_{i=0}^{r-1}\prod_{j\in\mathcal{N}_i}\mathcal{E}^{(i)}(z_j)\mathfrak{p}_{-\mu^i}(\diamondsuit_i)|0\rangle$ as a linear combination of $\mathfrak{p}_{-\nu}$'s with $\nu = (\nu^0,\ldots,\nu^{r-1}) \in \mathcal{P}_n(r)$. Also observe that

$$\langle \mathfrak{p}_{-\lambda}, \mathfrak{p}_{-\nu}\rangle = \prod_i \big\langle \mathfrak{p}_{-\lambda^i}(\diamondsuit_i)|0\rangle, \mathfrak{p}_{-\nu^i}(\diamondsuit_i)|0\rangle\big\rangle.$$

It follows from these observations and (5.2) that

$$\begin{aligned}
& G_{\lambda,\mu;r}(z_1,\ldots,z_N;k_1,\ldots,k_N)\\
&= \sum_{\mathcal{N}} F(\mathbf{z},\mathbf{k};\mathcal{N})\cdot\prod_{i=0}^{r-1}\left\langle \mathfrak{p}_{-\lambda^i}(\diamondsuit_i)|0\rangle, \prod_{j\in\mathcal{N}_i}\mathcal{E}^{(i)}(z_j)\mathfrak{p}_{-\mu^i}(\diamondsuit_i)|0\rangle\right\rangle\\
&= \sum_{\mathcal{N}}\left(F(\mathbf{z},\mathbf{k};\mathcal{N})\cdot\prod_{i=0}^{r-1} G_{\lambda^i,\mu^i;1}(rz_{l_{i,1}},\ldots,rz_{l_{i,n_i}})\right).
\end{aligned}$$

This completes the proof. $\square$

For $N = 1$ the above proposition reads

$$G_{\lambda,\mu;r}(z;k) \;=\; \sum_{i=0}^{r-1} e^{za(k,i)} G_{\lambda^i,\mu^i;1}(rz). \tag{5.3}$$

REMARK 5.2. When $r = 1$, the N-point function $G_{\lambda^i,\mu^i;1}(rz_1,\ldots,rz_N)$ of intersection numbers on Hilbert schemes of points on the affine plane has been studied in some detail in [**LQW2**]. It was shown in *loc. cit.* that the N-point functions for the affine plane has a precise connection with the N-pointed functions of disconnected Gromov-Witten invariants of $\mathbb{P}^1$ studied in [**OP**]. In addition, an explicit formula

for the 1-point function was given in Theorem 4.2, [**LQW2**]. When combined with (5.3), we obtain the 1-point function for a general r.

5.2. The τ-functions of intersection numbers on $\mathcal{M}(p,n)$. For $0 \le i \le r-1$, we introduce two sequences of indeterminates:

$$t_i = (t_{i,1}, t_{i,2}, \ldots), \qquad s_i = (s_{i,1}, s_{i,2}, \ldots).$$

Set $t = (t_0, \ldots, t_{r-1}), s = (s_0, \ldots, s_{r-1})$. Define the following half vertex operators:

$$\Gamma_{\pm}^{(i)}(t) = \exp\left(\sum_{m>0} \frac{1}{m} t_{i,m} \mathfrak{p}_{\pm m}(\diamondsuit_i)\right).$$

Define the numbers $n_{k;d}$, where $d \ge -1, 0 \le k \le r-1$, by the generating function

$$\sum_{d=-1}^{\infty} n_{k;d}\, z^d = \frac{e^{-kz}}{1-e^{-rz}}. \tag{5.4}$$

Introduce the following elements in $\mathbb{H}_n^{(p)}$, which are linear combinations of equivariant Chern characters of $L_k(p)^{[n]}$ (for $0 \le k \le r-1$ and $m \ge -1$):

$$\widetilde{\mathrm{ch}}_{k;m}^{[n](p)} = \sum_{d=-1}^{m+1} \left(n_{k;d}\,\mathrm{ch}_{k;m-d}^{[n](p)} - n_{k+1;d}\,\mathrm{ch}_{k+1;m-d}^{[n](p)}\right) \tag{5.5}$$

where we adopt the convention $\mathrm{ch}_{r;m-d}^{[n](p)} = \mathrm{ch}_{0;m-d}^{[n](p)}$. As we shall see in Proposition 5.3, these modified Chern characters admit explicit geometric constructions.

Given a multi-partition $\mu = (\mu^0, \cdots, \mu^{r-1})$ with $\mu^i = (\mu_1^i, \mu_2^i, \ldots)$, we define

$$t_{\mu^i}^{(i)} = t_{i,\mu_1^i} t_{i,\mu_2^i} \cdots, \qquad t_\mu = \prod_i t_{\mu^i}^{(i)},$$

and similarly define s_μ. Let $x_i = (x_{i,0}, x_{i,1}, x_{i,2}, \ldots)$, $0 \le i \le r-1$ and $x = (x_0, \ldots, x_{r-1})$, be some other sequences of indeterminates. We introduce the following generating function for the equivariant intersection numbers on $\mathcal{M}(p,n)$:

$$\tau(x,t,s,p) = \sum_n \sum_{\|\lambda\|=\|\mu\|=n} t_\lambda s_\mu \left\langle \lambda, \exp\left(\sum_{k=0}^{r-1}\sum_{m=0}^{\infty} x_{k,m} \widetilde{\mathrm{ch}}_{k;m}^{[n](p)}\right), \mu \right\rangle^{(p)}$$

By solving (4.7) we obtain for $0 \le k \le r-1$ that

$$\mathcal{E}^{(k)}(z) = \frac{1}{1-e^{-rz}}\left(e^{-kz}\mathfrak{H}_k(z) - e^{-(k+1)z}\mathfrak{H}_{k+1}(z)\right)$$

where we let $\mathfrak{H}_r(z) = \mathfrak{H}_0(z)$. Denote by $\mathcal{E}^{(k)}(z) = \sum_{m=-1}^{\infty} \mathcal{E}_{k;m} z^m$. Then,

$$\begin{aligned}\mathcal{E}_{k;m}(t^{n(p)}) &= \mathrm{Coeff}_{z^m}\left\{\mathcal{E}^{(k)}(z)(t^{n(p)})\right\}\\ &= \mathrm{Coeff}_{z^m}\left\{\frac{1}{1-e^{-rz}}\left(e^{-kz}\mathfrak{H}_k(z)(t^{n(p)}) - e^{-(k+1)z}\mathfrak{H}_{k+1}(z)(t^{n(p)})\right)\right\}\end{aligned}$$

where $t^{n(p)}$ denotes the image of t^n under the isomorphism $S_p : \mathbb{H}_n \to \mathbb{H}_n^{(p)}$. Combining this with Theorem 4.5, we obtain

$$\mathcal{E}_{k;m}(t^{n(p)}) = \mathrm{Coeff}_{z^m}\left\{\frac{e^{-kz}}{1-e^{-rz}}\widetilde{\mathfrak{G}}_k^{(p)}(z)(t^{n(p)}) - \frac{e^{-(k+1)z}}{1-e^{-rz}}\widetilde{\mathfrak{G}}_{k+1}^{(p)}(z)(t^{n(p)})\right\}.$$

By the definition of the operator $\widetilde{\mathfrak{G}}_k^{(p)}(z)$, (5.4) and (5.5), we get

$$\mathcal{E}_{k;m}(t^{n(p)}) = \widetilde{\mathrm{ch}}_{k;m}^{[n](p)}. \tag{5.6}$$

Proposition 5.3. *For $\alpha \in H_T^*(X)$, let $G_m^T(\alpha, n)$ be the $H_T^{2m}(X^{[n]})$-component of the class $\pi_{1*}\left(\mathrm{ch}^T(\mathcal{O}_{\mathcal{Z}_n}) \cup \pi_2^*\alpha \cup \pi_2^*\mathrm{td}^T(X)\right)$, where $\mathrm{td}^T(X)$ is the T-equivariant Todd class of X. Then, $\widetilde{\mathrm{ch}}_{k;m}^{[n](0)} = t^{n-m-1}G_{m+1}^T(\diamondsuit_k/r, n)$ for $m \ge -1$.*

Proof. First, applying the equivariant Grothendieck-Riemann-Roch Theorem [**EGr**] to $L_k^{[n]} = \pi_{1*}(\mathcal{O}_{\mathcal{Z}_n} \otimes \pi_2^* L_k)$, we have

$$\mathrm{ch}^T(L_k^{[n]}) = \pi_{1*}\left(\mathrm{ch}^T(\mathcal{O}_{\mathcal{Z}_n}) \cup \pi_2^*\mathrm{ch}^T(L_k) \cup \pi_2^*\mathrm{td}^T(X)\right).$$

It follows that

$$G_m^T(\mathrm{ch}^T(L_k), n) = \mathrm{ch}_m^T(L_k^{[n]}). \tag{5.7}$$

Next, recall that the operator $\mathfrak{G}_k(z) \in \mathrm{End}(\mathbb{H})$ is defined by the $\star$-product with the class $\sum_{m\ge 0} t^{n-m}\mathrm{ch}_m^T(L_k^{[n]})z^m$ in $\mathbb{H}_n$ for every n. We define the operator

$$\mathfrak{G}^{(\alpha)}(z) \in \mathrm{End}(\mathbb{H})$$

by the $\star$-product with $\sum_{\ell\ge 0} t^{n-\ell}G_\ell^T(\alpha, n)z^{\ell-1}$ in $\mathbb{H}_n$ for every n.

Note that $\diamondsuit_i/r \cup \diamondsuit_j/r = \delta_{i,j}\cdot t\ \diamondsuit_i/r$. By (4.1) and Lemma 2.1 (i), we have

$$\mathrm{ch}^T(L_k) = \sum_{i=0}^{k-1}\sum_{d=0}^{\infty} \frac{(k-r)^d t^{d-1}}{d!}\diamondsuit_i/r + \sum_{i=k}^{r-1}\sum_{d=0}^{\infty}\frac{k^d t^{d-1}}{d!}\diamondsuit_i/r. \tag{5.8}$$

Therefore, by (5.7), (5.8) and the linearity of $G_m^T(\alpha, n)$ on α, we have

$$\begin{aligned}
& \sum_{m=0}^{\infty} t^{n-m}\mathrm{ch}_m^T(L_k^{[n]})z^m \\
= & \sum_{m=0}^{\infty} t^{n-m}G_m^T(\mathrm{ch}^T(L_k), n)z^m \\
= & \sum_{m=0}^{\infty} z^m t^{n-m}\sum_{d=0}^{\infty}\left(\sum_{i=0}^{k-1}\frac{(k-r)^d}{d!}G_m^T(t^{d-1}\diamondsuit_i/r, n) + \sum_{i=k}^{r-1}\frac{k^d}{d!}G_m^T(t^{d-1}\diamondsuit_i/r, n)\right) \\
= & \sum_{m=0}^{\infty} z^m t^{n-m}\sum_{d=0}^{m}\left(\sum_{i=0}^{k-1}\frac{(k-r)^d t^{d-1}}{d!}G_{m-d+1}^T(\diamondsuit_i/r, n)\right. \\
& \qquad\qquad \left. + \sum_{i=k}^{r-1}\frac{k^d t^{d-1}}{d!}G_{m-d+1}^T(\diamondsuit_i/r, n)\right)
\end{aligned}$$

which, by setting $\ell = m - d + 1$, is equal to

$$\begin{aligned}
& \sum_{i=0}^{k-1}\sum_{d=0}^{\infty}\frac{(k-r)^d z^d}{d!}\sum_{\ell=0}^{\infty} z^{\ell-1}t^{n-\ell}G_\ell^T(\diamondsuit_i/r, n) \\
+ & \sum_{i=k}^{r-1}\sum_{d=0}^{\infty}\frac{k^d z^d}{d!}\sum_{\ell=0}^{\infty} z^{\ell-1}t^{n-\ell}G_\ell^T(\diamondsuit_i/r, n).
\end{aligned}$$

It follows from the definition of the operator $\mathfrak{G}$'s that

$$\mathfrak{G}_k(z) = \sum_{i=0}^{k-1} e^{(k-r)z} \mathfrak{G}^{(\diamondsuit_i/r)}(z) + \sum_{i=k}^{r-1} e^{kz} \mathfrak{G}^{(\diamondsuit_i/r)}(z). \tag{5.9}$$

Thus, $\mathfrak{G}^{(\diamondsuit_i/r)}(z) = \mathcal{E}^{(i)}(z) \mid_{\mathbb{H}}$ by applying Theorem 4.5, solving (4.7) for $\mathcal{E}^{(i)}(z)$ and solving (5.9) for $\mathfrak{G}^{(\diamondsuit_i/r)}(z)$. Combining this with (5.6), we obtain

$$\widetilde{\mathrm{ch}}_{k;m}^{[n](0)} = \mathcal{E}_{k;m}(t^n) = \mathrm{Coeff}_{z^m} \left\{ \mathcal{E}^{(k)}(z)(t^n) \right\} = \mathrm{Coeff}_{z^m} \left\{ \mathfrak{G}^{(\diamondsuit_i/r)}(z)(t^n) \right\}.$$

It follows from the definition of $\mathfrak{G}^{(\diamondsuit_i/r)}(z)$ that $\widetilde{\mathrm{ch}}_{k;m}^{[n](0)} = t^{n-m-1} G_{m+1}^T(\diamondsuit_k/r, n)$. $\square$

REMARK 5.4. We can enhance the ring isomorphism $\phi : \mathbb{H}_n \longrightarrow R(\Gamma_n)$ in Theorem 3.4 as follows. By comparing Proposition 3.2 and Lemma 4.4, we see that

$$\phi(\mathcal{E}^{(i)}(z) \cdot a) = \mathfrak{O}^{(\gamma_i)}(z) \cdot \phi(a).$$

It follows that

$$\phi\big(\widetilde{\mathrm{ch}}_{k;m}^{[n](0)}\big) = \Xi_n^m(\gamma_k).$$

Thus, as a counterpart of a result in [**Wa2**], the classes $\widetilde{\mathrm{ch}}_{k;m}^{[n](0)}$, where $0 \le m < n$ and $0 \le k \le r-1$, form a set of ring generators for $\mathbb{H}_n$.

THEOREM 5.5. (i) *Let $p \in P$. The τ-function $\tau(x,t,s,p)$ affords the following operator formulation:*

$$\left\langle |0\rangle, S_p^{-1} \prod_{k=0}^{r-1} \Gamma_+^{(k)}(t) \cdot \exp\left(\sum_{k=0}^{r-1} \sum_{m=0}^{\infty} x_{k,m} \mathcal{E}_{k;m} \right) \cdot \prod_{k=0}^{r-1} \Gamma_-^{(k)}(s)\, S_p\, |0\rangle \right\rangle$$

(ii) *Write $p = \sum_i n_i \diamondsuit_i$ with $n_i \in \mathbb{Z}$. Then,*

$$\tau(x,t,s,p) = \prod_{k=0}^{r-1} \tau(x_k, t_k, s_k, n_k)$$

Here $\tau(x_k, t_k, s_k, n_k)$ denotes

$$\left\langle |0\rangle, S_{\diamondsuit_k}^{-n_k} \Gamma_+^{(k)}(t) \cdot \exp\left(\sum_{m=0}^{\infty} x_{k,m} \mathcal{E}_{k;m} \right) \cdot \Gamma_-^{(k)}(s)\, S_{\diamondsuit_k}^{n_k}\, |0\rangle \right\rangle$$

and it satisfies the 2-Toda hierarchy of Ueno-Takasaki [**UT**].

PROOF. Note that

$$\Gamma_-^{(i)}(s) = \sum_{n \ge 0} \sum_{|\lambda^i| = n} t_{\lambda^i}^{(i)} \mathfrak{p}_{-\lambda^i}(\diamondsuit_i)$$

and $\Gamma_+^{(i)}(t)$ is the adjoint operator of $\Gamma_-^{(i)}(t)$. Now Part (i) follows from (5.6).

Observe that the bilinear form has the factorization property:

$$\langle [\lambda], [\mu] \rangle = \prod_{k=0}^{r-1} \langle [\lambda^k], [\mu^k] \rangle$$

for $\lambda = (\lambda^0, \cdots, \lambda^{r-1})$ and $\mu = (\mu^0, \cdots, \mu^{r-1})$. Now Part (ii) follows from the fact that the operators associated to different k in part (i) commute.

Note that the functions $\tau(x_k, t^k, s^k, n_k)$ can be interpreted as generating functions of the equivariant intersection numbers on Hilbert schemes of points on the affine plane (i.e. $r = 1$ in our case) studied in [**LQW2**]. It was shown in *loc. cit.* that $\tau(x_k, t^k, s^k, n_k)$ satisfies the 2-Toda hierarchy of Ueno-Takasaki [**UT**]. $\square$

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